Cohomology of finite groups: abstract nonsense

For scheduling reasons, this lecture is being recorded on Thursday, January 14. For those watching live, as usual there will be 30 minutes of office hours immediately afterward.

HW 9 is due today. Please ask in advance (i.e., today) if you need an extension.

HW 10 is posted. It is due Thursday, January 21.

No lectures or office hours on Monday, January 18.

G-modules

G-modules form an abelian category

Let G be a (finite) group

A (right) **G-module** is an abelian group A equipped with a G-action (on right).

$$G \times A \longrightarrow A \qquad (g,a) \mapsto a^g$$

$$(a+b)^g = a^g + b^g$$

$$a^{g_1 g_2} = (a^{g_1})^{g_2}$$

i.e. a right $\underbrace{\mathbb{Z}[G]}_{\text{group algebra}}$-module

A **homomorphism** of G-modules is a homomorphism $\phi: A \to B$ of underlying abelian groups which is G-equivariant. i.e. $\phi(a^g) = \phi(a)^g$.

Invariants (and coinvariants)

M = G-module.

$M^G = \{ m \in M : m^g = m \quad \forall g \in G \}$

G-invariants = maximal subobject of M which is G-fixed

$M_G = M /$ submodule generated by $m^g - m$ $\forall g \in G$ $\forall m \in M$

= G-coinvariants

= maximal quotient of M which is G-fixed.

Invariants, coinvariants, and exact sequences

Let $0 \to M' \xrightarrow{\text{inj}} M \xrightarrow{\text{surj}} M'' \to 0$ be an exact sequence of G-modules

(i.e. kernel of each map = image of previous map)

Then

$$0 \to M'^G \to M^G \to M''^G \cancel{\to 0}$$

is exact

i.e. G-invariants functor from G-modules to abelian groups is left ~~exact~~ exact

Ex: $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

a sequence of G-modules where $G = \mathbb{Z}/p\mathbb{Z}$

$a^g = a(1+pg)$.

Similarly, G-coinvariants functor is right exact.

Where we are headed: derived functors

We are looking for: there exist functors

$H^i(G, \cdot) : \{G\text{-modules}\} \to \{\text{abelian groups}\}$ $\quad i = 0, 1, 2, \ldots$

right derived functors

$H^0(G, A) = A^G$

such that given an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

get the exact sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \xrightarrow{\delta^0}$$
$$\to H^1(G, M') \to H^1(G, M) \to H^1(G, M'') \xrightarrow{\delta^1}$$
$$\to H^2(G, M') \to \cdots$$

Injective G-modules (dual notion: projective)

A G-module M is injective if for every inclusion $A \subset B$ of G-modules, every G-module homomorphism $A \to M$ can be extended to B.

$$\begin{array}{ccc} A & \longrightarrow & M \\ \downarrow & \nearrow & \\ B & & \end{array}$$

Prop Every G-module is a subobject of some injective G-module.

(That is, the category of G-modules has enough injectives.)

Pf For G = {e}, injective = divisible (i.e. $\forall m \in M$, $n \in \mathbb{N}_{>0}$, $\exists m'$ s.t. $nm' = m$)

In general, $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], N)$ where $M \to N$ is injection of ab groups with N divisible.

The category of G-modules has enough injectives

Injective resolutions

For M a G-module, an injective resolution of M is an exact sequence

$$0 \to M \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

of G-modules where $I^0, I^1, \ldots$ are injective.

Definition of the derived functors

Apply G-invariants: I get a sequence

$$0 \to M^G \to I^{0G} \xrightarrow{d^0} I^{1G} \xrightarrow{d^1} \cdots$$

exact

of abelian groups which is a complex ($d^{i+1} \circ d^i = 0 \ \forall i \geq 0$) & not necessarily exact.

Define $H^i(G, M) = \ker(d^i) / \operatorname{im}(d^{i-1})$

(where d^{-1} = zero map into I^0,

so $H^0(G, M) = M^G$

Claim: $H^i(G, M)$ depends only on M (up to unique isomorphism)

Functoriality

Let $f: M \to N$ be a morphism of G-modules

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[r] \arrow[d, "f^1"] & \cdots \\ 0 \arrow[r] & N \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots \end{tikzcd}$$

Given two injective resolutions

(use injectivity property to fill in vertical maps).

Apply G-invariants:

$$\begin{tikzcd} 0 \arrow[r] & M^G \arrow[r] \arrow[d] & I^{0G} \arrow[r] \arrow[d] & I^{1G} \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & N^G \arrow[r] & J^{0G} \arrow[r] & J^{1G} \arrow[r] & \end{tikzcd}$$

induce maps $H^i(f): H^i(G, M) \to H^i(G, N)$.

<u>Claim</u>: this depends only on f, not on other choices

Key tool: the five lemma

Lemma: Consider a diagram of R-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

then f is an isomorphism $\iff$ f', f'' are isomorphism's.

more generally, have something similar 4

$$\begin{tikzcd}
M^0 \arrow[r] \arrow[d] & M^1 \arrow[r] \arrow[d] & M^2 \arrow[r] \arrow[d] & M^3 \arrow[r] \arrow[d] & M^4 \arrow[d] \\
N^0 \arrow[r] & N^1 \arrow[r] & N^2 \arrow[r] & N^3 \arrow[r] & N^4
\end{tikzcd}$$

} exact rows

Key tool: the snake lemma

Lemma Consider commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Then there exists an exact sequence

$$0 \to \ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'') \to 0.$$

← connecting "homomorphism" "snake"

Define δ by "diagram chase":

Acyclic resolutions

A G module M is acyclic if $H^i(G, M) = 0$ $\forall\, i > 0$.

(e.g. if M injective, then acyclic:

use $0 \to M \xrightarrow{id} M \to 0 \to 0 \to \cdots$)

Key fact: if

$0 \to M \to I^0 \xrightarrow{d^0} I^1 \to \cdots$ is an acyclic resolution

(i.e. an exact sequence with $I^0, I^1, \ldots$ acyclic)

then $H^i(G, M) = \ker(d^{iG}) / \mathrm{im}(d^{i-1\,G})$

$$I^{iG} \xrightarrow{d^{iG}} I^{(i+1)G}.$$