Cohomology of finite groups: abstract nonsense

For scheduling reasons, this lecture is being recorded on Thursday, January 14. For those watching live, as usual there will be 30 minutes of office hours immediately afterward.

HW 9 is due today. Please ask in advance (i.e., today) if you need an extension.

HW 10 is posted. It is due Thursday, January 21.

No lectures on the last Monday, January 18.
G-modules

Let $G$ be a (finite) group $G$.

A $G$-module is a left $A$-module $A$ equipped with a $G$-action (on the right).

$G \times A \to A$ 

$(g, a) \mapsto a^g$

\[(a + b)^g = a^g + b^g\]
\[g \cdot a = (g, 1) a\]

A homomorphism of $G$-modules is a homomorphism $\phi: \Lambda \to \Lambda$ which is $G$-equivariant, i.e., $\phi(\alpha \cdot g) = \phi(\alpha) e^g$. 
Invariants (and coinvariants)

\[ M = G - \text{module} \]

\[ M^G = \{ m \in M : m^g = m \quad \forall g \in G \} \]

\[ G - \text{invariants} = \text{maximal submodule of } M \]

\[ G - \text{invariants} \text{ which is } G - \text{fixed} \]

\[ M_c = M / \text{submodule generated by } mGm \]

\[ = G - \text{coinvariants} \]

\[ = \text{maximal } \gamma \text{ where } \gamma \text{ is } G - \text{fixed} \]
Invariants, coinvariants, and exact sequences

Let \( 0 \to M' \to M \to M'' \to 0 \) be an exact sequence of \( R \)-modules.

(i) Kernel of each map = image of previous map

Then \( 0 \to M' \to M \to M'' \to 0 \) is exact.

Ex. \( 0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0 \)

a sequence of \( \mathbb{Z} \)-modules, where \( \mathbb{Z} = \mathbb{Z}/4\mathbb{Z} \)

is exact.

Similarly, coinvariant functor is right exact.
Where we are headed: derived functors

We are looking for the cohomology $H^i(G, \mathcal{O})$ of $G$-modules $\mathcal{O}$ in degree $i$.

$H^i(G, \mathcal{O})$ is a right derived functor of $(-) \cdot \mathcal{O}$.

$H^0(G, \mathcal{A}) = \mathcal{A}$

such that given an exact sequence

$0 \to M' \to M \to M'' \to 0$

the long exact sequence

$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'')$ $\to \cdots$

$\cdots H^1(G, M') \to H^1(G, M) \to H^1(G, M'')$ $\to \cdots$

$\cdots H^2(G, M') \to H^2(G, M) \to H^2(G, M'')$ $\to \cdots$
Injective G-modules

A G-module is injective if for every inclusion \( A \subseteq B \) of G-modules, every G-module homomorphism \( A \to B \) can be extended to \( B \).

Prop. Every G-module is a subobject of some injective G-module.

That is, the category of G-modules has enough injectives.

PE (for G-\( \mathbb{G} \)), injective = divisible (in the sense of existence, \( M \to \text{Hom}_G(2C6, M) \) where \( M \) is a projective divisible module).
The category of G-modules has enough injectives
Injective resolutions

For $M$ a $b$-module, an injective resolution of $M$ is an exact sequence

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$$

where $I^0, I^1, I^2, \ldots$ are injective modules.
Definition of the derived functors

Apply $6$-inverses: let a sequence

\[ 0 \to M \xrightarrow{e} \cdots \xrightarrow{d^i} I^i \to \cdots \]

which is a complex of a left exact

A not necessarily exact.

Define $H^i(\mathcal{G}, M) = \ker(d^i) / \ker(d^{i-1})$

where $d^{-1} = 0$ and into $I^0$.

So $H^0(\mathcal{G}, M) = M^0$.

Claim: $H^i(\mathcal{G}, M)$ depends only on $\mathcal{G}$ (up to isomorphism).
Functoriality

Let $f: M \to N$ be an $\mathcal{O}$-morphism of $\mathcal{O}$-modules.

\[
\begin{array}{c}
0 \to M' \\
\downarrow f ' \\
0 \to N'
\end{array}
\]

Given two $\mathcal{O}$-modules $M' \to N'$, consider the following sequence:

\[
\begin{array}{c}
0 \to M \\
\downarrow f \\
0 \to N
\end{array}
\]

For any $\mathcal{O}$-invariant $\alpha$:

\[
\begin{array}{c}
0 \to M \\
\downarrow f \\
0 \to N
\end{array}
\]

induce maps $1^n(f): 1^n(M) \to 1^n(N)$. (Claim: this depends only on $f$, not on other choices)
Key tool: the five lemma

**Lemma:** Consider a diagram of modules

\[
0 \to M' \to M \to M'' \to 0
\]

If
\[
0 \to N' \to N \to N'' \to 0
\]

Then \( f \) is an isomorphism \( \iff f' \) is an isomorphism.

More generally, have something similar if

\[
\begin{array}{c}
M_1 \to M_2 \to M_3 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
M'_1 \to M'_2 \to M'_3
\end{array}
\]

\[
\begin{array}{c}
N_1 \to N_2 \to N_3 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
N'_1 \to N'_2 \to N'_3
\end{array}
\]

is exact.
Key tool: the snake lemma

\[ \begin{array}{ccccccccc}
0 & \to & M' & \to & M & \to & N & \to & 0 \\
\downarrow \quad & & \downarrow \quad & & \downarrow \quad & & \downarrow \quad & \quad & \quad \\
0 & \to & \ker f' & \to & \ker f & \to & \ker f'' & \to & 0 \\
\end{array} \]

The snake consists in an exact sequence

\[ 0 \to \ker f' \to \ker f \to \ker f'' \to 0 \]

Denote \( \delta \) by "diagram chase".
Acyclic resolutions

A complex $M$ is **acyclic** if $H^i(C, M) = 0$.

(e.g. if $M$ injective, then acyclic: $A \implies 0$).

Key note: if $0 \to M \to I \to \cdots$ is an acyclic resolution,

then $H^1(C, M) = \ker (d^1) / \text{im } (d^0)$.

$$
\begin{array}{c}
\text{Acyclic resolutions} \\
A \text{ complex } M \text{ is } \text{acyclic if } H^i(C, M) = 0 \\
\text{(e.g. if } M \text{ injective, then acyclic: } A \implies 0). \\
\text{Key note: if } 0 \to M \to I \to \cdots \text{ is an acyclic resolution,} \\
\text{then } H^1(C, M) = \ker (d^1) / \text{im } (d^0). \\
\end{array}
$$