Cohomology of finite groups: concrete nonsense

As always, see the course web site https://math.ucsd.edu/~kedlaya/math204b/ for recordings of previous lectures.

PS9 has been collected. Please feel free to discuss it on Zulip.

PS10 is due Thursday, January 21.

PS11 will be posted later this week. It will be due Thursday, January 28.

For those interested in doing a final project, I have posted some topic suggestions to Zulip. These are not exhaustive!

Reminder: G-modules and their invariants

G = (finite) group — right $\mathbb{Z}[G]$-module

A right G-module is an abelian group M equipped with an additive right G-action

$$(g, m) \longmapsto m^g \qquad (m^{g_1})^{g_2} = m^{g_1 g_2}$$

These form an abelian category.

$$M^G = \{ m \in M : m^g = m \ \forall g \in G \}$$

$$\{\text{right } G\text{-modules}\} \longrightarrow \{\text{abelian groups}\}$$

Reminder: group cohomology as derived functors

The functor $M \mapsto M^G$ is left exact, so it admits right derived functors

$$H^i(G, \cdot) : \{G\text{-modules}\} \to \{\text{abelian groups}\}$$

$$H^0(G, M) = M^G$$

Given an exact sequence $0 \to M' \to M \to M'' \to 0$ of G-modules, get a long exact sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \xrightarrow{\delta^0} H^1(G, M') \to H^1(G, M) \to H^1(G, M'') \xrightarrow{\delta^1} H^2(G, M') \to \cdots$$

Acyclic resolutions

How do compute $H^i(G, M)$?

$$\left[\begin{array}{l} 0 \to M \to N^0 \to N^0/M \to 0 \\ 0 \to N^0/M \to N^1 \to N^1/\ldots \to 0 \\ \ldots \end{array}\right.$$

Say $0 \to M \to N^0 \xrightarrow{d} N^1 \xrightarrow{d} \ldots$ is an exact sequence

where $H^i(G, N^j) = 0$ $\forall$ $i > 0$, $j \geq 0$

(acyclic resolution)

Then consider the sequence

$$0 \to N^{0G} \xrightarrow{d^{0G}} N^{1G} \xrightarrow{d^{1G}} N^{2G} \to \ldots$$

then $H^i(G, M) = \ker(d^{iG}) / \operatorname{im}(d^{(i-1)G})$

(e.g. if N^j are injective G-modules, then acyclic.)

Induced G-modules

$H \leq G$ inclusion of groups

$\mathrm{Res}^G_H : \{G\text{-modules}\} \to \{H\text{-modules}\}$

restriction

$$\boxed{\mathrm{Ind}^G_H M = M \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]}$$ induced G-module.

$m \otimes [g]$

// induction of a H-module

$\{ \phi : G \to M : \phi(gh) = \phi(g)^h \}$

with action $\quad \phi^g(g') = \phi(gg')$

$\phi_{m,g} : g' \to \begin{cases} m^{gg'} & \text{if } gg' \in H \\ 0 & \text{otherwise.} \end{cases}$

Frobenius reciprocity and Shapiro's lemma

M - G-module

N - H-module

$$\mathrm{Hom}_G(M, \mathrm{Ind}_H^G N) = \mathrm{Hom}_H(\mathrm{Res}_H^G M, N)$$

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G N, M) = \mathrm{Hom}_H(N, \mathrm{Res}_H^G M)$$

i.e. Res_H^G and Ind_H^G are adjoint functors in both directions.

assumes G finite.

this holds more generally

Frobenius reciprocity and Shapiro's lemma

Shapiro's lemma: $H \subseteq G$, N = H-module,

$$H^i(G, \mathrm{Ind}_H^G N) \cong H^i(N, H).$$

∃ a canonical isomorphism.

Pf: OK for $i = 0$.

- Ind_H^G is exact ("$\mathbb{Z}[G]$ is a free left $\mathbb{Z}[H]$-module")
- If N is injective, then so is $\mathrm{Ind}_H^G N$ (by adjoint property).

Cor: If M is an induced G-module (i.e. induced from $\{e\} = H \subseteq G$), then M is acyclic

Homogeneous cochains

G = finite group
M = G-module

$i = 0, 1, \ldots$

$$N^i = \{ \phi : G^{i+1} \to M \}$$

with G action

$$\phi^g(g_0, \ldots, g_i) = \phi(g_0 g^{-1}, \ldots, g_i g^{-1})^g$$

This is induced: $N^i = \mathrm{Ind}_{\{e\}}^G N_0^i$

$$N_0^i = \{ \phi \in N^i : \phi(g_0, \ldots, g_i) = 0 \text{ where } g_0 \neq e \}$$

A resolution by homogeneous cochains

$d^i : N^i \to N^{i+1}$

$(d^i\phi)(g_0, \dots, g_{i+1}) = \sum_{j=0}^{i+1} (-1)^j \phi(g_0, \dots, \hat{g_j}, \dots, g_{i+1})$.

- $\boxed{d^{i+1} \circ d^i = 0.}$
- $0 \to M \to N^0 \xrightarrow{d^0} N^1 \to \dots$ is exact.

$M = N^{-1}$

- d^i is G-equivariant.

$\Rightarrow 0 \to M \to N^0 \to N^1 \to \dots$ is an acyclic resolution!

Fun with H^1

A homogeneous 1-cochain (which maps to zero under d^1) satisfies

$$\rho(g) = \phi(e, g)$$

$$0 = (d^1\phi)(e, h, gh)$$

$$= \phi(h, gh) - \phi(e, gh) + \phi(e, h)$$

$$= \phi(e, g)^h - \rho(gh) + \rho(h)$$

$$= \rho(g)^h - \rho(gh) + \rho(h)$$

i.e., ρ is a crossed homomorphism.

$\phi = d^0(\psi)$

ϕ is zero in H^1 iff

$\rho(g) = \psi(e)^g - \psi(e)$

principal crossed homomorphism

Fun with H^1

$H^1(G, M)$ = principal homogeneous spaces of M (A)

= have G-action & M-action

$\forall a \in A$, $M \to A$ $\quad m \mapsto m(a)$

is a bijection.

Fun with H^2

$H^2(G, M)$ = "classifies" short-exact sequences

$$\begin{tikzcd} 1 \arrow[r] & M \arrow[r] & E \arrow[r] & G \arrow[r] \arrow[l, bend right, "s"'] & 1 \end{tikzcd}$$

of groups with fixed G-action on M

$$\begin{tikzcd} 1 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & M \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$