Cohomology and homology of finite groups

HW11 is now posted.

Addendum: the restriction/induction adjunction

Let $H \subseteq G$ be an inclusion of finite groups.
Let M be a G-module, let N be an H-module.
Then $\mathrm{Hom}_G(M, \mathrm{Ind}_H^G N) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G M, N)$

$\mathrm{Hom}_G(\mathrm{Ind}_H^G, N) \cong \mathrm{Hom}_H(N, \mathrm{Res}_H^G M)$

In particular, if $N = \mathrm{Res}_H^G M$, get maps

$M \to \mathrm{Ind}_H^G \mathrm{Res}_H^G M$ $\qquad$ $\mathrm{Ind}_H^G \mathrm{Res}_H^G M \to M$

$m \longmapsto \sum_i m^{g_i} \otimes [g_i^{-1}]$ $\qquad$ $\sum_g m_g \otimes [g] \longmapsto \sum_{g \in G} (m_g)^g$

where g_i runs over left cosets of H in G. ← requires $(G:H) < \infty$

Extended functoriality for cohomology

G, G' finite groups $M = G$-module, $M' = G'$-module

Have also $\alpha : G' \longrightarrow G$ group hom, $\beta : M \longrightarrow M'$ homomorph of ab groups

with property that $\beta(m^{\alpha(g')}) = \beta(m)^{g'}$ homomorphism of G'-modules

$\forall m \in M, g' \in G'$

$$H^i(G, M) \xrightarrow{\mathrm{Res}_\alpha} H^i(G', M) \xrightarrow{H^i(\beta)} H^i(G', M')$$

Extended functoriality for cohomology: examples

1. $\alpha : G \to G$ $\quad g \mapsto h^{-1} g h$ $\quad h \in G$

   $\beta : M \to M$ $\quad m \mapsto m h$

   then $H^i(G, M) \to H^i(G, M)$ are identity.

2. $\alpha : G' \to G$, $\quad \beta = \mathrm{id}_M$

   $\mathrm{Res} : H^i(G, M) \to H^i(G', M)$

3. $H \subseteq G$, $\quad \mathrm{Ind}_H^G \mathrm{Res}_H^G M \to M$

corestriction

$\mathrm{Cor} : H^i(H, M) \xrightarrow[\text{Shapiro's}]{\sim} H^i(G, \mathrm{Ind}_H^G M) \to H^i(G, M)$

$\mathrm{Cor} \circ \mathrm{Res}$ induced by $\underbrace{M \to \mathrm{Ind}_H^G M \to M}_{\text{mult by } [G:H]}$

Extended functoriality for cohomology: examples

Cor: $H^i(G, M)$ is killed by $\#G$.

(with $H = \{e\}$)

$H \lhd G$ $\qquad \alpha: G \to G/H$

M = G-module $\qquad \beta: M^H \hookrightarrow M$

M^H = G/H-module

$$\mathrm{Inf}: H^i(G/H, M^H) \to H^i(G, M)$$

inflation

(Hochschild-Serre spectral sequence)

The augmentation ideal and coinvariants

$\mathbb{Z}[G]$ = group algebra of G

$\cup\!|$

$\underline{I_G}$ = <u>augmentation ideal</u> = $\ker\{\mathbb{Z}[G] \to \mathbb{Z}\}$

$\{n_g [g]\} \mapsto \{m_g$

= ideal gen by $[g]-1$ $\forall g \in G$.

M = a G-module

$M_G = M/MI_G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$

$= M/\{mg - m : m \in M, g \in G\}$ ideal gen by

module of coinvariants

$M \twoheadrightarrow M_G$

Homology via projective resolutions

$M \to M_G$ is right exact: $0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow$ $M'_G \to M_G \to M''_G \to 0$

A G-module N is projective if $\forall$ surj. $M \to M'' \to 0$, any hom $N \to M''$ lifts to some $N \to M$.

$$\begin{tikzcd} M \arrow[r] & M'' \arrow[r] & 0 \\ N \arrow[u, dotted] \arrow[ur] & & \end{tikzcd}$$

Using projective resolutions $\underbrace{\cdots \to N_1 \to N_0 \to M \to 0}_{\text{exact}}$ (N_i projective) can define $H_i(G, M)$ (kernel/image in $\cdots \to N_{1G} \to N_{0G} \to 0$)

$H_0(G, M) = M_G$

Note: every free $\mathbb{Z}(G)$-module is projective!

Fun with H_1

$M = \mathbb{Z}$ trivial G-module.

$$0 \to I_G \to \mathbb{Z}(G) \to \mathbb{Z} \to 0$$

projective

long exact sequence

$$\cdots \; H_i(G, \mathbb{Z}G) \to H_i(G, \mathbb{Z}) \to H_{i-1}(G, I_G) \to \cdots$$

$$H_i(G, \mathbb{Z}G) = 0$$

$$\underbrace{H_1(G, \mathbb{Z}G)}_{=0} \to H_1(G, \mathbb{Z}) \xrightarrow{\simeq} I_G/I_G^2 \xrightarrow{0} \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z} \to 0$$

$G^{ab} \cong I_G/I_G^2 \qquad g \longrightarrow [g]-1$

$= G/[G,G]$ (exercise)

$H \leq G$

$$H_1(G, \mathbb{Z}) \cong G^{ab}$$

The norm map

G = finite group

Let M = G-module

$$\mathrm{Norm}_G : M \to M \qquad \mathrm{Norm}_G(m) = \sum_{g \in G} m^g$$

$$H_0(G, M) = M_G \to M^G = H^0(G, M)$$

$$\mathrm{Norm}_G(m^h - m) = \sum_{g \in G} m^{hg} - \sum_{g \in G} m^g = 0$$

The Tate cohomology groups

$$H_T^i(G, M) = \begin{cases} H^i(G, M) & i > 0 \\ M^G / \mathrm{Norm}_G(M) & i = 0 \\ \ker(\mathrm{Norm}_G) / M I_G & i = -1 \\ H_{-i-1}(G, M) & i < -1 \end{cases}$$

The long exact sequence for Tate cohomology

For any SES $0 \to M' \to M \to M'' \to 0$ of G-modules get

long exact sequence

$$\cdots \to H_T^{i-1}(G, M'') \to H_T^i(G, M') \to H_T^i(G, M) \to H_T^i(G, M'')$$

$$\to H_T^{i+1}(G, M') \cdots \quad \cdots \to H_T^{-1}(G, M'')$$

$$\to H_T^{-1}(G, M') \to H_T^{-1}(G, M) \to H_T^{-1}(G, M'')$$

$$\to H_T^0(G, M') \to H_T^0(G, M) \to H_T^0(G, M'')$$

$$\to H^1(G, M') \to H^1(G, M) \to H^1(G, M'') \to \cdots$$

The cyclic case: Tate's periodicity theorem

Now G = finite cyclic $= \langle g \rangle$

Thm: $\exists$ canonical isomorphism $\forall i \in \mathbb{Z}$
functorial $\forall M$.

$$H^i_T(G, M) \xrightarrow{\simeq} H^{i+2}_T(G, M)$$

Proof of Tate's theorem

B/c $G = \langle g \rangle$ is cyclic

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

is exact.

$$[h] \mapsto 1$$

$$[h] \mapsto [hg] - [h]$$

$$1 \mapsto \sum_{h \in G} [h]$$

induced modules

$\Leftarrow$ any cyclic

$$\hat{H}^i_T(G, \mathbb{Z}) \cong \hat{H}^{i+1}_T(G, I_G) \cong \hat{H}^{i+2}_T(G, \mathbb{Z})$$

The Herbrand quotient

G cyclic

$$0 \to M' \to M \to M'' \to 0$$

long exact sequence

$$\begin{tikzcd}
 & H^1_T(G,M) \arrow[r] & H^{-1}_T(G,M'') \arrow[dr] & \\
H^1_T(G,M') \cong H^1_T(G,M') \arrow[ur] & & & H^0_T(G,M') \arrow[dl] \\
 & H^0_T(G,M'') \arrow[ul] & H^0_T(G,M) \arrow[l] &
\end{tikzcd}$$