Herbrand quotients; profinite groups

3.2 3.4

Reminder: Tate cohomology groups

G = finite group $\qquad M$ = (right) G-module

$M^G = \{m \in M : m^g = m \ \forall g \in G\}$ $\quad$ G-invariants

$M_G = M / \{m^g - m : m \in M, g \in G\}$ $\quad$ G-coinvariants

$H^i(G,M)$ $\quad i \geq 0$, $H^0(G,M) = M^G$

$H_i(G,M)$ $\quad i \geq 0$, $H_0(G,M) = M_G$

$$H^i_T(G,M) = \begin{cases} H^i(G,M) & i > 0 \\ M^G / \mathrm{Norm}_G M & i = 0 \\ \ker(\mathrm{Norm}_G) / M I_G & i = -1 \\ H_{-i-1}(G,M) & i < -1 \end{cases}$$

$\mathrm{Norm}_G : M_G \to M^G$

$\mathrm{Norm}_G(m) = \sum_{g \in G} m^g$

Reminder: the cyclic case $= \langle g \rangle$

Now assume G is finite cyclic.

Tate's theorem: In this case $\exists$ isomorphism

$$\hat H^i_T(G,M) \cong \hat H^{i+2}_T(G,M)$$

functorial once you fix a generator.

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

$$1 \mapsto \sum [h] \qquad\qquad [h] \mapsto 1$$

$$h \in G \quad [h] \mapsto [hg] - [h]$$

$$0 \to \mathbb{Z} \to \underbrace{\mathbb{Z}[G]} \to I_G \to 0 \qquad 0 \to I_G \to \underbrace{\mathbb{Z}[G]} \to \mathbb{Z} \to 0$$

trivial Tate groups

The Herbrand quotient of a G-module

"Euler characteristic"

G = finite cyclic group

M = G-module.

Define the Herbrand quotient of M as

$$h(M) = \frac{\# H^0_T(G,M)}{\# H^{-1}_T(G,M)} \quad \left\{ \text{provided that these are finite} \right.$$

If $0 \to M' \to M \to M'' \to 0$ is exact, then so is

$$\begin{tikzcd}
 & H^{-1}_T(G,M) \arrow[r] & H^{-1}_T(G,M'') \arrow[dr] & \\
H^{-1}_T(G,M') \arrow[ur] & & & H^0_T(G,M') \arrow[dl] \\
 & H^0_T(G,M'') \arrow[ul] & H^0_T(G,M) \arrow[l] &
\end{tikzcd}$$

these finite $\Longrightarrow$ this; and this are finite

$$\Longrightarrow \boxed{h(M) = h(M')\,h(M'')}$$

Example: a finite G-module

$G = \langle g \rangle$

If M is finite then $h(M) = 1$

$$0 \to M^G \to M \to M \to M_G \to 0$$

$$m \mapsto m^g - m$$

$$\Rightarrow \# M^G = \# M_G$$

$$0 \to H^{-1}_T(G,M) \to M_G \xrightarrow{\mathrm{Norm}_G} M^G \to H^0_T(G,M) \to 0$$

$$\Rightarrow \# H^0_T(G,M) = \# H^{-1}_T(G,M).$$

Example: an extension of local fields

K = finite ext of $\mathbb{Q}_p$

L/K = finite cyclic Galois ext

$G = \mathrm{Gal}(L/K)$ $\quad M = L^*$

Here $H^1_T(G, M) = H^1(G, M) = 0$ (Hilbert 90)

and $H^0(G, M) = K^*$ $\quad$ e.g. $K = \mathbb{Q}_3$

but $H^0_T(G, M)$ = finite. $\quad$ $\cancel{L} = \mathbb{Q}_3(\sqrt{-1})$

$= K^*/\mathrm{Norm}_{L/K}(L^*)$ $\quad$ $\mathrm{Norm}_{L/K}(L^*) = 3^{2\mathbb{Z}} \times \mathcal{O}_K^*$

$(\mathrm{Norm}_{L/K} : \mathcal{O}_L^* \to \mathcal{O}_K^*$ surjective$)$

Extended functoriality for homology/Tate groups

Let $\alpha: G' \to G$ be a hom of finite groups & let $\beta: M \to M'$ be a hom of G'-modules

where $M = G$-mod, $M' = G'$-mod

we have induced maps $H^i(G, M) \to H^i(G', M')$

but $\nexists$ for $H_i(G, M) \to H_i(G, M')$

w/ $H^i_T(G, M) \to H^i_T(G, M)$

unless get a map $M_G \to M'_{G'}$ (e.g. α is on it, surjective)

Changing gears...

Profinite groups: topological definition

A profinite group is a topological group G which is compact Hausdorff, and its topology has a neighborhood basis of identity consisting of (~~normal~~ open) subgroups (of finite index)

caution: in general, not every finite-index subgroup has to be open!

e.g. $\mathbb{Z}_p$

Profinite groups: interpretation as inverse limits

Say I = poset $\{G_i\}_{i \in I}$ a family of finite groups

for each pair $(i, j) \in I \times I$ with $i \geq j$

have a surjective map $G_i \to G_j$, also assume:

- I is filtered: any finite subset is dominated by some element?

- $G_i \to G_j \to G_k = G_i \to G_k$.

Then can form $G = \varprojlim_{i \in I} G_i$. This carries structure of a profinite group (each G_i has discrete top, G subspace top from $\prod_{i \in I} G_i$)

Some simple examples

$$\mathbb{Z}_p, \quad \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \overset{\text{CRT}}{\cong} \prod_p \mathbb{Z}_p$$

$$GL_m(\mathbb{Z}_p), \quad GL_m(\hat{\mathbb{Z}}) \cong \widehat{GL_m(\mathbb{Z})}$$

$$= \varprojlim_n GL_m(\mathbb{Z}/n\mathbb{Z})$$

For any group G, <u>profinite completion</u>

$$\hat{G} := \varprojlim_H G/H$$

where H runs over normal subgroups of finite index.

Galois groups as profinite groups

The Galois correspondence for infinite extensions

L/K is an infinite Galois extension of fields (e.g. $L = K^{sep}$)

then $\mathrm{Gal}(L/K) = \varprojlim \mathrm{Gal}(M/K)$

$\quad \leftarrow M/K \subset L/K$ finite Galois subextension

$$\begin{array}{ll} M_1 & \mathrm{Gal}(M_1/K) \\ | & \quad\downarrow \\ M_2 & \mathrm{Gal}(M_2/K) \\ | & \\ K & \end{array}$$

e.g. $K = \mathbb{F}_q$, $L = \overline{K}$

$\mathrm{Gal}(L/K) \cong \widehat{\mathbb{Z}}$

(procyclic)

subfields of L/K correspond to closed subgroups of $\mathrm{Gal}(L/K)$

$G \longmapsto \mathrm{Fix}\, G \qquad M/K \to \mathrm{Gal}(L/M)$

Next time: cohomology of profinite groups

Explain how to define cohomology of a profinite group G acting on a suitable topological G-module.