Cohomology of profinite groups; overview of local CFT

HW 11 due Thursday, January 28.

HW 12 to be posted later today.

Policy change: I've lowered the target for homework to 5 out of 9 (from 6 out of 9), maintaining the options to combine partial results across assignments and to make up one full assignment by doing a final project.

Riddle: National Public Radio just posted a news story about (a new book about) Sir Nicholas Winton, who rescued over 600 children in Prague from the advancing Nazi regime in the 1930s. This story has a remarkable connection to number theory, specifically *twin primes*. What is the connection?

Reminder: profinite groups (and Galois groups)

G = a topological group is <u>profinite</u> if

$G = \varprojlim_{i \in I} G_i$ G_i = finite groups I = poset

$i \geq j$: $G_i \twoheadrightarrow G_j$

(open sets in G are generated by $\pi_i^{-1}(S_i)$ $\pi_i : G \to G_i$, $S_i \subseteq G_i$)

<u>examples</u> $\mathbb{Z}_p$, $\widehat{\mathbb{Z}}$, $GL_n(\mathbb{Z}_p)$, $GL_n(\widehat{\mathbb{Z}})$ Galois field

or $\mathrm{Gal}(L/K)$ where L is any <u>algebraic</u> extension of K.
(e.g. $L = K^{sep}$)

$= \varprojlim \mathrm{Gal}(M/K)$
M/K finite extensions in L/K.

e.g. $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$ $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ = something explicit.

Neukirch-Schmidt-Wingberg

Discrete G-modules

G = profinite group

A G-module is a topological abelian group M

+ a continuous G-action $(G \times M \to M$ continuous$)$

A discrete G-module means M has discrete topology

Continuity of action map means: $\forall m \in M$.

$\mathrm{Stab}_G(m)$ is open. i.e. $M = \bigcup M^H$

$H \subseteq G$ open (normal)

e.g.

The key class of examples

K = field

L/K algebraic field extension

(e.g. $L = K^{sep}$)

$$M = L^* \qquad G = \mathrm{Gal}(L/K)$$

$$= \bigcup_{\substack{L' \subset L/K \\ \text{finite}}} L'^*$$

Inflation homomorphisms

G = profinite group
M = discrete G-module

$H_1 \subseteq H_2 \subseteq G$ $\quad H_1, H_2$ open normal in G.

$$\mathrm{Inf}: H^i(G/H_2, M^{H_2}) \longrightarrow H^i(G/H_1, M^{H_1})$$

finite quotients of G

$$\left(\begin{array}{l} G/H_1 \twoheadrightarrow G/H_2 \\ M^{H_2} \to M^{H_1} \end{array} \right)$$

Cohomology via inflation

e.g. $H^1(\mathrm{Gal}(L/K), L^*) = 0$
by Hilbert–Noether Theorem 90

Then

$$H^i(G, M) \cong \varinjlim H^i(G/H, M^H)$$

$H \trianglelefteq G$ open normal

(using inflation)
(warning: inflation maps need not be injective

(Reminder: $\varinjlim$ means take disjoint union,
divide by an equivalence relation:
$\alpha_1 \in H^i(G/H_1, M^{H_1})$ and $\alpha_2 \in H^i(G/H_2, M^{H_2})$
are equivalent if they have same image in some $H^i(G/H_3, M^{H_3})$.

An alternate approach: continuous cochains

can also form complex of continuous cochains

$$G^i \longrightarrow M$$

(when M discrete, any such cochain factors through $(G/H)^i$ for some $H \subset G$ open normal)

Changing gears...

The goal: the local reciprocity map

"local field" = finite extension of $\mathbb{Q}_p$ for some p

(or perhaps of $\mathbb{F}_p((t))$)

K = local field $\quad K^{ab}$ = maximal abelian extension of K

__Thm__ There is a unique map $\phi_K : K^* \to \mathrm{Gal}(K^{ab}/K)$ ← __local reciprocity map__

s.t.

1) For any uniformizer ~~w~~ π of K and any finite __unramified__ extension L/K, $\phi_K(\pi)$ acts on L as Frobenius.

2) For any L/K finite abelian, $K^* \xrightarrow{\phi_K} \mathrm{Gal}(K^{ab}/K)$

$$\begin{array}{ccc} K^* & \xrightarrow{\phi_K} & \mathrm{Gal}(K^{ab}/K) \\ \downarrow & & \downarrow \\ K^*/\mathrm{Norm}_{L/K}(L^*) & \xrightarrow{\sim} & \mathrm{Gal}(L/K). \end{array}$$

↑ norm residue symbol

Local reciprocity and local Kronecker-Weber

What should be the map for $K = \mathbb{Q}_p$?

$$K^{ab} = \bigcup_n \mathbb{Q}_p(\zeta_n) = K_1 K_2$$

$$K_1 = \bigcup_n \mathbb{Q}_p(\zeta_{p^n})$$

$$K_2 = \bigcup_n \mathbb{Q}_p(\zeta_{p^n-1})$$

$$\mathrm{Gal}(K^{ab}/K) \cong \mathrm{Gal}(K_1/K) \times \mathrm{Gal}(K_2/K)$$

$$\mathrm{Gal}(K_1/K) \cong \mathbb{Z}_p^\times$$

$$\mathrm{Gal}(K_2/K) = \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$$

Local reciprocity and local Kronecker-Weber

KW: $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^\times = \hat{\mathbb{Z}}^\times$

$$\cong \mathbb{Z}_p^* \times \prod_{\ell \neq p} \mathbb{Z}_\ell^*$$

p^e

LKW: $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \cong \mathbb{Z}_p^* \times \hat{\mathbb{Z}}$ $\quad e$

Candidate for local reciprocity map is

$$\mathbb{Q}_p^* = \mathbb{Z}_p^* \times \mathbb{Z} \to \mathbb{Z}_p^* \times \hat{\mathbb{Z}}.$$

Use this picture to check condition (2) of local reciprocity.

The local existence theorem

- For L/K finite (not necessarily abelian)

a) $\mathrm{Nm}_{L/K}\, L^*$ is finite index in K^*.

b) Conversely, every finite-index subgroup of K^* arises in this way, and actually the L/K <u>abelian</u>.

The norm limitation theorem

if L/K finite, then

$$\mathrm{Norm}_{L/K} L^* = \mathrm{Norm}_{M/K} M^*$$

where M/K is maximal abelian subextension of L/K.

(reminder: $\mathrm{Gal}(L/K)$ always soluble, so $M=K$ iff $L=K$.)