Overview of local class field theory continued

HW 12 has been posted (and slightly updated this morning).

Answer to Wednesday's riddle: one of the children saved by Sir Nicholas Winton was Heini Halberstam, who became an analytic number theorist. The *Elliott-Halberstam conjecture* plays a crucial role in the work of Goldston-Pintz-Yıldırım, Zhang, Maynard, Polymath, et al. on partial results towards the twin prime conjecture: it is proven that there are infinitely many pairs of prime numbers that differ by at most 250.

The local reciprocity law

\[ K = \text{Gal}(k \otimes \mathbb{Q}) \]

There is a unique group homomorphism

\[ \phi_k : k^* \rightarrow \text{Gal}(k^{ab}/k) \]

(local reciprocity map, norm residue symbol) such that:

1. for any unramified \( \mathfrak{p} \in \mathcal{O}_k \), \( \phi_k(\mathfrak{p}) \) projects \( \mathbb{Q}_k \left( \frac{k^{ab}}{k} \right) \) to \( \mathbb{Z} \) to \( \mathbb{F}_p \).
2. if \( \mathfrak{p} \) is unramified in \( \mathcal{O}_k \) and \( \mathcal{P} \subseteq \mathcal{P} \) is an extension \( \mathbb{L}/k \), the composition

\[ k^* \xrightarrow{\phi_k} \text{Gal}(k^{ab}/k) \]

induces an isomorphism

\[ k^*/(\text{Norm}_{\mathbb{L}/k} k^*) \approx \text{Gal}(k^{ab}/k) \]
The local existence theorem

- For $L/k$ finite (not necessarily cyclic), $N_{L/k}[L^*]$ is open subgroup of $L^*$, and the index $[L^*: N_{L/k}[L^*]]$ depends continuously on $L$. When $k^* = O^*$ is compact, the index is finite.

- Conversely, any open subgroup of $L^*$ of finite index in $L^*$ is normal in $L^*$ for some finite extension $L/k$. 
Aside: explicit existence via Lubin-Tate formal groups

Let $k \subseteq k'$ be fields.

$$K_{a,b} = K_{\text{unr}} \otimes K_1$$

where $K_1 = \text{Cryp}(1)$, the ring of formal power series in $x$ with $x$ nilpotent.

E.g., if $k = \mathbb{Q}_p$, $\pi = p$, then $K_{a,b} = \mathbb{Q}_p((\pi_p))$. Let $\rho = \text{Det}(1-g_0)$.

Then $K_{a,b}$ can be constructed explicitly using Lubin-Tate formal groups.
The norm limitation theorem

Let \( L/K \) be any finite extension.

Let \( M/K \) be a maximal abelian subextension.

Then \( \text{Norm}_{L/K} L^* = \text{Norm}_{M/K} M^* \).

Easy to show that \( \text{Norm}_{L/K} L = \text{Norm}_{M/K} M \).
An upgraded reciprocity law

\[ \frac{\mathbf{K}}{\mathbf{K}^*} \text{ for } \mathbf{K} \text{ abelian} \]

\[ \mathbf{K}^*/\mathbf{K} \xrightarrow{\sim} \mathbf{G}(\mathbf{K}) = \mathbf{G}^{\mathbf{K}} \]

\[ \text{Th} \quad \text{There are also isomorphisms} \]

\[ H^i_{\mathbf{T}}(\mathbf{G}, \mathbf{Z}) \xrightarrow{\sim} H^{i+2}_{\mathbf{T}}(\mathbf{G}, \mathbf{L}^*) \]

(\text{cup and cap})
Plan A: the local invariant map

There exists an isomorphism:

$$
\varphi: H^2(\mathbb{Q}(1)(\mathbb{Q}/\mathbb{Q}), \mathbb{Q}) \cong H^2(\mathbb{Q}(1)(\mathbb{Q}/\mathbb{Q}), \mathbb{Q})
$$

More precisely, by 2/1, we have:

$$
\varphi: \mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}
$$

and these are isomorphic with injection.
Aside: Brauer groups

\[ H^2(\text{Aut}(\mathbb{K}/K), \mathbb{K}^*) \simeq \text{Br}(K) \]

\[ \text{Br}(K) : \{ \text{someomorphism classes of finite dim. of K with center K} \} \]

\[ D_1 \circ D_2 = D \quad \text{where } D_1, D_2 \in \text{Aut}(D) \]

\[ \text{Br}(\mathbb{K}) = \{ 1, \mathbb{R} \} \subset \text{Aut}(\mathbb{K}) \simeq \mathbb{R}/2\mathbb{Z} \]
A(nother) theorem of Tate

Then
\[ G = \text{any Hecke group} \]
\[ M = \overline{G} \text{ mod } \mathbb{N} \]

Suppose:

for every subgroup \( H \leq G \),
\[ 1-I^1(H, M) = 0 \quad \forall \]
\[ M^2(H, \eta) \text{ is cyclic of order } \# H \]

Then get isomorphisms
\[ H^1(G, \mathbb{Z}) \cong H^2(G, M) \]
these are canonical once you fix a generator of \( H^2(G, M) \).
Plan B: abstract class field theory

\[ \forall A \text{ with: } \forall H^0(\text{Gal}(L/K), \mathbb{L}^*) \rightarrow C(L:K) \]
\[ \forall H^{-1}(\text{Gal}(L/K), \mathbb{L}^*) \rightarrow 1 \]

This is enough to recover local CFT!

More precisely, given a field \( K \), a continuous \( \mathbb{Q} \)-module \( A \), \( \mathcal{E} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), a signature \( \phi \) and a homomorphism \( \phi \rightarrow \mathcal{E} \) \( \text{unramified} \) \( \text{valuation} \) (\'unramified\) (\'valuation\)

will set...
Plan B: abstract class field theory

\[ \mathbb{C}/K \cong \mathbb{C}/K \]

\[ A_n = A_{\text{Gal}(\overline{K}/K)} \quad A_2 = A_{\text{Gal}(K/L)} \]

\[ A = K^* \]

\[ \lambda : \text{Gal}(\overline{K}/K) \to \text{Gal}(K^{\text{nr}}/K) \]

\[ \forall : A^* = \mathbb{K}^* \quad \mu_1 : \mathbb{Z} \to \mathbb{Z} \rightarrow \mathbb{Z} \]
Preview of next time: $H^2$ computations

Cool for next time: compute

$H^2(L/K) = H^2(\text{Gal}(L/K), L^*)$

for $L/K$ finite extension (where $L/K$ is finite).

Do the unramified case by explicitly computing $H^0(L/K) = L^*$

using Hecke operators.
Et cetera

- $L/K$ cyclic,

\[ \mathbb{Z}^A + H^2(L/K) \cong H^0(L/K) \]

again using Hochschild theory.

- General case

by induction with minimal case.

\[ L \supset K \implies (m \otimes K) = CL \otimes K \]