Overview of local class field theory continued

HW 12 has been posted (and slightly updated this morning).

Answer to Wednesday's riddle: one of the children saved by Sir Nicholas Winton was Heini Halberstam, who became an analytic number theorist. The *Elliott-Halberstam conjecture* plays a crucial role in the work of Goldston-Pintz-Yıldırım, Zhang, Maynard, Polymath, et al. on partial results towards the twin prime conjecture: it is proven that there are infinitely many pairs of prime numbers that differ by at most 250.

Photo from Halberstam's obituary in *The Guardian*.

The local reciprocity law

K = finite ext of $\mathbb{Q}_p$

Thm There is a unique homom $\phi_K : K^* \to \mathrm{Gal}(K^{ab}/K)$ (local reciprocity map, norm residue symbol) such that:

a) for any uniformizer $\pi \in K$, $\phi_K(\pi)$ projects in $\mathrm{Gal}(K^{unr}/K) \cong \hat{\mathbb{Z}}$ to Frobenius.

b) for any finite abelian extension L/K, the composition

$$\begin{array}{ccc} K^* & \xrightarrow{\phi_K} & \mathrm{Gal}(K^{ab}/K) \\ \downarrow & & \downarrow \\ K^*/\mathrm{Norm}_{L/K} L^* & \overset{\sim}{\dashrightarrow} & \mathrm{Gal}(L/K) \end{array}$$

factors through an isom

The local existence theorem

~~locally compact of~~ alg

- For L/K finite (not necessarily abelian), $\mathrm{Norm}_{L/K} L^*$ is open subgroup of finite index in K^*.

  (topology on K^* is subspace topology from $K^* \hookrightarrow K \times K$, $x \longmapsto (x, x^{-1})$)

  i.e. topology induced by iso $K^* \cong \mathcal{O}_K^* \times \pi^{\mathbb{Z}}$

  discrete

  $\dots \quad \pi^{-1}\mathcal{O}_K^* \quad \mathcal{O}_K^* \quad \pi\mathcal{O}_K^* \quad \dots$

- Conversely, any open subgroup of finite index in K^* is $\mathrm{Norm}_{L/K} L^*$ for some finite abelian extension L/K.

Aside: explicit existence via Lubin-Tate formal groups

Fix K, $\pi \in K$ uniformizer

$$K^{ab} = K^{unr} \circ K_\pi$$

K_π = compositum of finite abelian extensions L/K with $\pi \in \mathrm{Norm}_{L/K} L^*$

$(\sim K^* = \mathcal{O}_K^* \cdot \pi^{\mathbb{Z}})$

(e.g. if $K = \mathbb{Q}_p$, $\pi = p$, $K_\pi = \bigcup_n \mathbb{Q}_p(\zeta_{p^n})$ $\quad p = \mathrm{Norm}_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}(1 - \zeta_{p^n})$)

Then K_π can be constructed explicitly using Lubin-Tate formal groups

The norm limitation theorem

Thm Let L/K be any finite extension

Let M/K be maximal abelian subextension

then $\mathrm{Norm}_{L/K} L^* = \mathrm{Norm}_{M/K} M^*$

$\subseteq$ easy $\qquad \supseteq$

$\mathrm{Norm}_{L/K} = \mathrm{Norm}_{M/K} \circ \mathrm{Norm}_{L/M}$

An upgraded reciprocity law

L/K finite abelian

$$K^*/\mathrm{Norm}_{L/K} L^* \xrightarrow[\phi_K]{\sim} \mathrm{Gal}(L/K) = G = G^{ab}$$

$$= H_T^0(G, L^*) \qquad\qquad = H_1(G, \mathbb{Z})$$

$$= H_T^{-2}(G, \mathbb{Z})$$

<u>Thm</u> There are also isomorphisms

$$H_T^i(G, \mathbb{Z}) \xrightarrow{\sim} H_T^{i+2}(G, L^*)$$

(cup product)

Plan A: the local invariant map

Thm $\exists$ canonical isomorphisms:

$$H^2(\mathrm{Gal}(K^{unr}/K), K^{unr*}) \underset{\text{inflation}}{\cong} H^2(\mathrm{Gal}(\overline{K}/K), \overline{K}^*) \underset{\text{inv}}{\cong} \mathbb{Q}/\mathbb{Z} = \bigcup_n \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

More precisely, for L/K of degree n,

$$\mathrm{inv}_{L/K} : H^2(\mathrm{Gal}(L/K), L^*) \cong \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

and these are compatible with inflation.

Aside: Brauer groups

$$H^2(\mathrm{Gal}(\bar{K}/K), \bar{K}^\times) \cong Br(K)$$ Brauer group of K

$Br(K) = \{$ isomorphism classes of finite dim'l K division algebras with center $K \}$

$D_1 \circ D_2 = D$ where $D_1 \otimes_K D_2 \cong M_r(D)$

$$Br(\mathbb{R}) = \{\mathbb{R}\} \cup \{\mathbb{H}\} \cong \mathbb{Z}/2\mathbb{Z}$$

A(nother) theorem of Tate

<u>Thm</u> G = finite group
M = G-module

<u>Suppose:</u> for every subgroup $H \leq G$,
$H^1(H, M) = 0$ and
$H^2(H, M)$ is cyclic ~~of~~ order $\# H$

<u>Then</u> get isomorphisms $H^i_T(G, \mathbb{Z}) \cong H^{i+2}_T(G, M)$
these are canonical once you ~~fix~~ fix
a generator of $H^2(G, M)$.

e.g. $G = \mathrm{Gal}(L/K)$
$M = L^*$
$H = \mathrm{Gal}(L/L')$

Plan B: abstract class field theory

Start with: $\# H^0_T(\mathrm{Gal}(L/K), L^*) = (L:K)$

for L/K cyclic: $\# H^{-1}_T(\mathrm{Gal}(L/K), L^*) = 1$

This is enough to recover local CFT!

More precisely, given: a field K, $G = \mathrm{Gal}(\overline{K}/K)$.

a continuous G-module A,

a surjective continuous hom $G \to \hat{\mathbb{Z}}$ ("unramified")

and a hom $v: A^G \to \hat{\mathbb{Z}}$ ("valuation")

will get ...

Plan B: abstract class field theory

$\forall L/K$ finite

— Canonical isom $\mathrm{Gal}(L/K)^{ab} \cong A_K / \mathrm{Norm}_{L/K} A_L$

$A_K = A^{\mathrm{Gal}(\bar{K}/K)}$ $\quad A_L = A^{\mathrm{Gal}(\bar{K}/L)}$

apply this with $\quad A = \bar{K}^*$

$$d : \mathrm{Gal}(\bar{K}/K) \to \mathrm{Gal}(K^{unr}/K) \cong \hat{\mathbb{Z}}$$

$$v : A^G = K^* \xrightarrow[\mathrm{val}]{} \mathbb{Z} \to \hat{\mathbb{Z}}.$$

Preview of next time: H^2 computations

Goal for next time: compute

$$H^2(L/K) = H^2(\mathrm{Gal}(L/K), L^*)$$

for L/K finite extension (where $K/\mathbb{Q}_p$ is finite)

- do the unramified case by explicitly computing $H^0_T(L/K) = \dfrac{K^*}{\mathrm{Norm}\, L^*}$ using Herbrand quotient.

Et cetera

- L/K cyclic,
  see $\# H^2(L/K) = \# H^0_T(L/K)$
  again using the Herbrand quotient.

- General case
  by comparison with unramified case.

$$
\begin{tikzcd}
L \arrow[dr, dash] & & m \\
& K \arrow[ur, dash, "\text{unr}"'] &
\end{tikzcd}
\qquad (m : K) = (L : K)
$$