Cohomology of local fields

Reminder: if you are watching the live lecture, you can use the Miro link from the web site to see all of my boards at once. (If you are watching the recording, you can download the PDF instead.)

CFT 4.2

Summary Local invariant map

Theorem 90 $H^1(L/K) = 0$ $\hat{=}$ $H^1(\mathrm{Gal}(L/K), L^*)$

Need to show:

for L/K finite Galois extension with $(K:\mathbb{Q}_p) < \infty$.

$H^2(L/K) =$ cyclic of order $(L:K)$

more precisely, it "is" $\frac{1}{(L:K)}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$

and this is compatible with field extensions:

M/K finite Galois extension containing L,

$$\begin{array}{ccc} H^2(L/K) & \xrightarrow{\text{Inf}} & H^2(M/K) \\ \cap & & \swarrow \\ & \mathbb{Q}/\mathbb{Z} & \end{array}$$

commutes $\Rightarrow H^2(\overline{K}/K) \cong \mathbb{Q}/\mathbb{Z}$.

Extensions of finite fields

Prop: If L/K an extension of finite fields, $\mathrm{Norm}_{L/K} : L^* \to K^*$ is surjective.

Pf - Direct calculation (exercise).

Or L/K cyclic Galois extension and $h(L^*) = 1$ b/c L^* is finite.

and $H^1_T(L/K) = H^1(L/K) = \{1\}$ (Thm 90)

so $H^0_T(L/K) = \{1\}$.

The norm on units in an unramified extension

Prop L/K finite unramified extension, then $\mathrm{Norm}_{L/K} : \mathcal{O}_L^* \to \mathcal{O}_K^*$ is surjective.

Pf Say $u \in \mathcal{O}_K^*$. By previous slide, $\exists v_0 \in \mathcal{O}_L^*$ s.t. $\mathrm{Norm}_{L/K}(v_0) \equiv u \bmod \pi_L$. $\leftarrow$ uniformizer

Now construct $v_i \equiv 1 \bmod \pi_L^i$ s.t. $i = 1, 2, \ldots$

$u_i := u / \mathrm{Norm}(v_0 \ldots v_i) \equiv 1 \bmod \pi_L^{i+1}$

(reduces to Trace: $\mathcal{O}_L/\pi_L \to \mathcal{O}_K/\pi_K$ is surjective) true b/c residue field ext is separable).

Herbrand quotient of the units: unramified case

L/K be finite unramified $\quad \boxed{0 \to \mathcal{O}_L^* \to L^* \xrightarrow{v} \mathbb{Z} \to 0}$

($\Rightarrow$ cyclic)

<u>Prop</u> $H_T^i(\mathrm{Gal}(L/K), \mathcal{O}_L^*) = 1 \quad \forall i \in \mathbb{Z}$. $\qquad G = \mathrm{Gal}(L/K)$

<u>Pf</u> By periodicity check

$i = 0$ – prev slide.

$i = 1$. $H^1(\mathrm{Gal}(L/K), L^*) = 1$ by Thm 90

~~but~~ since L/K <u>unramified</u>, (Same use π_K instead of π_L)

$L^* = \mathcal{O}_L^* \cdot \pi_K^{\mathbb{Z}}$ splitting of G-modules,

$H^i(\mathrm{Gal}(L/K), \mathcal{O}_L^*) \oplus H^i(\mathrm{Gal}(L/K), \mathbb{Z}) = 1$

Cohomology of the units: unramified case

$G = \mathrm{Gal}(L/K)$

$h(\mathcal{O}_L^*) = 1$, so $h(L^*) = h(\mathbb{Z}) = \#\underbrace{H^0_T(G,\mathbb{Z})}_{\#\mathbb{Z}/\mathrm{Nm}\,\mathbb{Z} = [L:K]} / \#\underbrace{H^1_T(G,\mathbb{Z})}_{=1}$

$= [L:K]$

$\Rightarrow$ since $H^1_T(L/K) = 1$,

$\Rightarrow |\# H^2(L/K)| = [L:K]$

$$1 \to H^0_T(G, L^*) \cong \underbrace{H^0_T(G,\mathbb{Z})}_{\mathbb{Z}/[L:K]\mathbb{Z}} \to \cancel{1}$$

$\cong$

$H^2_T(G, L^*) = H^2(G, L^*)$

The local invariant map: unramified case

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

acyclic (exercise)

so

$$H^2(G, L^*) \cong H^1(G, \mathbb{Q}/\mathbb{Z})$$ cyclic with a distinguished generator

$$\cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$$

evaluate at Frobenius $\cong \frac{1}{(L:K)}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$

$$\Rightarrow H^2(K^{unr}/K) \cong \mathbb{Q}/\mathbb{Z}$$

Herbrand quotient of the units: cyclic case

$G = \mathrm{Gal}(L/K)$

L/K finite cyclic, but possibly ramified.

$H^1(L^*) = 0$ by Thm 90

Lemma: $\exists$ open, Galois-stable subgroup W of $\mathcal{O}_L^*$ s.t. $H^i(G, W) = 0$ $\forall i > 0$.

Pf: First note: $\exists$ open Galois-stable subgroup V of $\mathcal{O}_L$ (additive) s.t. $H^i(G, V) = 0$ $\forall i > 0$. (normal basis theorem) V is induced.

Now exponentiate! $\exp(x) = \sum_{i=0} x^i/i!$ has positive radius of convergence

$W = \exp(V)$

Herbrand quotient of the units: cyclic case

$h(W) = 1$ and $h(\mathcal{O}_L^*/W) = 1$ b/c $\mathcal{O}_L^*/W$ finite.

$\Rightarrow h(\mathcal{O}_L^*) = 1$

$\Rightarrow h(L^*) = h(\mathbb{Z}) = [L:K]$

$\Rightarrow$ since $H^1(L/K) = 1$, $\# H_T^2(L/K) = [L:K]$

$= \# H^2(L/K)$

but not yet known that this group is cyclic!

[Aside: this is enough for "abstract CFT"]

The inflation-restriction exact sequence

Let G be a finite group, $H \trianglelefteq G$ normal, M - G-module.

- $0 \to H^1(G/H, M^H) \xrightarrow{\text{Inf}} H^1(G, M) \xrightarrow{\text{Res}} H^1(H, M)$ is exact. (pf: write in terms of crossed homs.)
- If $H^i(H, M) = 0$ for $i = 1, \ldots, r-1$, then $0 \to H^r(G/H, M^H) \xrightarrow{\text{inf}} H^r(G, M) \xrightarrow{\text{res}} H^r(H, M)$ is exact. (Dimension shifting. $0 \to M \to \text{Ind}_1^G \text{Res}_1^G M \to N \to 0$)

Need $r = 2$ in what follows. $i = 1$ vanishing will be Thm 90

Spectral sequence

An upper bound on H^2

Cor: $M/L/K$ is a tower of finite extensions of fields

$$0 \longrightarrow H^2(L/K) \xrightarrow{\text{Inf}} H^2(M/K) \xrightarrow{\text{Res}} H^2(M/L)$$

is exact.

Now suppose K finite ext of $\mathbb{Q}_p$.

$\Rightarrow \# H^2(M/K) \leq \# H^2(L/K) \# H^2(M/L)$

$\Rightarrow$ by induction ($\mathrm{Gal}(L/K)$ is solvable)

reduction to cyclic case

$\Rightarrow$ for any finite extension L/K of fin ext of $\mathbb{Q}_p$

$\# H^2(L/K) \leq [L:K]$

H^2 by comparison to the unramified case

L/K any finite Galois extension $(K:\mathbb{Q}_p) < \infty$

pick M/K unramified of same degree.

$$
\begin{array}{ccccccc}
 & & & & H^2(M/K) & & \\
 & & & & \downarrow \inf & \searrow & \\
0 & \longrightarrow & H^2(L/K) & \xrightarrow{\inf} & H^2(ML/K) & \longrightarrow & H^2(ML/L)
\end{array}
$$

$\Rightarrow H^2(M/K)$ cyclic order $= [L:K]$

WTS: this is zero.

order $\leq [L:K]$