The local invariant map; Tate's theorem

HW 13 is posted. It is due Thursday, February 11.

Reminder: state of the H^2 computation

Last time: compute $H^2(L/K) = H^2(\mathrm{Gal}(L/K), L^\times)$,
where $L/K/\mathbb{Q}_p$ are finite extensions, L/K Galois

desired answer: cyclic of order $[L:K]$.

- if L/K unramified, OK.
- if L/K cyclic, checked order.
- if L/K general ($\Rightarrow$ solvable), checked order $\leq [L:K]$

by inflation-restriction

Comparison with an unramified extension

Let M/K be the unramified extension of degree $[L:K]$

Then cyclotome$(M:K) = H^2(M/K)$

$\downarrow \mathrm{Inf}$ is this zero?

exact $0 \longrightarrow H^2(L/K) \xrightarrow{\mathrm{Inf}} H^2(ML/K) \xrightarrow{\mathrm{Res}} H^2(ML/L)$

If diagonal is zero, then a generator of $H^2(M/K)$

lands in $\mathrm{Ker}(\mathrm{Res}) = H^2(L/K) \hookrightarrow H^2(L/K) = H^2(M/K)$

inside $H^2(ML/K)$

Computation via H^0_T

M/K unramified $(M:K) = (L:K)$

Let U/K be maximal unramified extension within L/K

$\mathrm{Gal}(ML/L) \cong \mathrm{Gal}(M/U)$

$$e = e(L/U)$$
$$f = f(L/K)$$

$$\begin{tikzcd}[column sep=small]
 & ML \arrow[dl, dash, "e"'] \arrow[dr, dash, "e"] & \\
L \arrow[dr, dash, "e"'] & & M \arrow[dl, dash, "e"] \\
 & U &
\end{tikzcd}$$

linearly disjoint

$$\begin{tikzcd}
H^2(M/K) \arrow[r, "\mathrm{Res}"] \arrow[d, phantom, "\cong" rotate=90] & H^2(M/U) \arrow[r, "M^\times \hookrightarrow (ML)^*"'] \arrow[d, phantom, "\cong" rotate=90] & H^2(ML/L) \arrow[d, phantom, "\cong" rotate=90] \\
H^0_T(M/K) \arrow[r] \arrow[d, equal] & H^0_T(M/U) \arrow[r] \arrow[d, equal] & H^0_T(ML/L) \arrow[d, equal] \\
K^*/\mathrm{Norm}_{M/K} M^* \arrow[r] & U^*/\mathrm{Norm}_{M/U} U^* \arrow[r] & L^*/\mathrm{Norm}_{ML/L}(ML)^*
\end{tikzcd}$$

$=$ cyclic, order f, gen by $\pi_K = \text{unit} \cdot \pi_L^e$

cyclic of order e gen by π_L

~~Computation via H^0_T~~

comment about local reciprocity

For each finite Galois extension L/K, we have $H^2(L/K) = $ cyclic of order $(L:K)$

previous arguments shows that

$$H^2(K^{unr}/K) \xrightarrow{\inf} H^2(\bar{K}/K) \text{ is an isomorphism}$$

$$\cong \mathbb{Q}/\mathbb{Z}$$

And now... statement of Tate's theorem

<u>Thm</u> Let G be a finite group.
Let M be a G-module.

Suppose $\forall H \subseteq G$ subgroups, $H^1(H, M) = 0$
& $H^2(H, M) = $ cyclic of order $\# H$.

Then $\exists$ isomorphisms

$$H^i_T(G, \mathbb{Z}) \xrightarrow{\sim} H^{i+2}_T(G, M)$$

which are determined by a choice of generator of $H^2(G, M)$

e.g. K = local field
L/K = finite Galois ext.
$M = L^*$ $G = \mathrm{Gal}(L/K)$
$H^*(H, M)$
$= H^*(L/?)$

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take $i = -2$: $H^{-2}_T(G, \mathbb{Z}) \cong H^0_T(G, L^*)$

$G^{ab} = H_1(G, \mathbb{Z}) \longleftarrow = K^* / N_{L/K} L^*$

Proof of Tate's theorem: a key exact sequence

Let γ be a generator of $H^2(G, M)$ represented by a cocycle $\phi : G^3 \to M$

Define a G-module $M(\phi)$ (splitting module) s.t.

- $M(\phi)$ is acyclic for Tate cohomology
- it fits into an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & M(\phi) \arrow[r] \arrow[dr] & \mathbb{Z}[G] \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
& & 0 \arrow[r] & I_G \arrow[u] \arrow[r] & 0 &
\end{tikzcd}$$

$\Longrightarrow$

$$H^i_T(G, \mathbb{Z}) \xrightarrow{\sim} H^{i+1}_T(G, I_G) \xrightarrow{\sim} H^{i+2}_T(G, M)$$

Definition of the splitting module

$$M[\phi] = M \oplus \bigoplus_{g \in G - \{e\}} \mathbb{Z} x_g$$

with G-action

$$x_h^g = x_{hg} - x_g + \phi(e, g, hg)$$

where $x_e = \phi(e, e, e)$.

Properties of ϕ imply that this gives a G-action:

$$\phi(g_0 g, g_1 g, g_2 g) = \phi(g_0, g_1, g_2)^g$$

$$\phi(g_1, g_2, g_3) - \phi(g_0, g_2, g_3) + \phi(g_0, g_1, g_3) - \phi(g_0, g_1, g_2) = 0.$$

By construction, class γ maps to zero in $H^2(G, M[\phi])$:

use $\rho(e, g) = x_g$.

Towards acyclicity of the splitting module

$$0 \to M \to M(\phi) \to I_G \to 0$$

$M \to \mathbb{Z}$

$x_g \to [g] - 1$

$0 \to I_G \to \underset{\text{acyclic}}{\mathbb{Z}(G)} \to \mathbb{Z} \to 0$

restrict to H, take long exact sequence:

zero by design.

$$\cancel{0 = H^1(H, M)} \to H^1(H, M(\phi)) \to H^1(H, I_G) \to H^2(H, M) \to H^2(H, M(\phi)) \to 0$$

0

forced to be zero.

$H^0_T(H, \mathbb{Z}) = \mathbb{Z}/\#H\mathbb{Z}$: cyclic of order $\#H$

$\boxed{\to H^2(H, I_G) \cong H^1_T(H, \mathbb{Z}) = \mathrm{Hom}(H, \mathbb{Z}) = 0}$

Lemma: a shortcut to acyclicity (positive indices)

Lemma: Let G be a finite group, M = G-module. Suppose $H^i(H, M) = 0$ for all $H \subseteq G$ subgroup, all $i \in \{1,2\}$.

Then $H^i_T(G, M) = 0$ $\forall i$.

Suppose for the moment that G is solvable. Induction on $|G|$: pick $H \trianglelefteq G$ s.t. G/H cyclic nontrivial. Given $H^i(H, M) = 0$. Inflation-restriction $\Rightarrow$

$$0 \to H^i(G/H, M^H) \xrightarrow{\inf} H^i(G, M) \xrightarrow{\text{res}} H^i(H, M) = 0$$

check that $G/H \circlearrowright M^H$ satisfies hypotheses: reduce to cyclic case! ($\Leftarrow$ periodicity)

Lemma: a shortcut to acyclicity (negative indices)

For negative indices, do dimension shifting using

$$0 \longrightarrow N \longrightarrow \underline{\mathrm{Ind}_1^G \mathbb{Z} \otimes M} \longrightarrow M \longrightarrow 0$$

$$H_T^i(M) \cong H_T^{i+1}(N) \quad (n = 0, \ldots)$$

argue by induction on n that result holds

for indices $-n+1, \ldots, \infty$

Lemma: a shortcut to acyclicity (nonsolvable case)

if G not solvable,

let p = be any prime $\quad G_p = p$-Sylow subgroup (solvable)

claim:

$$H^i(G,M) \xrightarrow{\text{res}} H^i(G_p, M) \xrightarrow{\text{cor}} H^i(G,M)$$

injective on p-primary part

point: cor $\circ$ res $= [G : G_p]$ = coprime to p

So p-primary part of $H^i(G,M)$ vanishes for all p (~~the~~ groups ~~are~~ take torsion)