The local invariant map; Tate's theorem

HW 13 is posted. It is due Thursday, February 11.
Reminder: state of the $H^2$ computation

Last time: conjecture $H^2(L/K) = H^2(L/K)$,

where $L/K / Q_p$ are finite extensions,

and hence: $H^1_{rig}(L/K)$.

- if $L/K$ unramified, OK.
- if $L/K$ cyclic, checked OK.
- if $L/K$ general (=) solvable, checked

by $\mathbf{H}^1_{f,rig}$-restriction.
Comparison with an unramified extension

Let $M/K$ be the unramified extension of degree $L$. Then $\operatorname{cyc} (\text{unram}) = H^2 (M/K)$. If $\chi$ is trivial, then $\operatorname{cyc} (\text{unram}) = 0$. The exact sequence:

$$0 \rightarrow H^2 (L/K) \rightarrow H^2 (M/L) \rightarrow 0.$$ 

If dimension 1, then the generator is $H^2 (M/L)$.
Computation via $H^0_T$

Let $L/K$ be a normal integral extension with $L/K$

$\text{Gal}(L/K) \cong \text{Gal}(M/L)$

$c = c(L/k)$

$f = f(L/K)$

\[
\begin{array}{c}
\text{H}^2(M/K) \to \text{H}^2(L/K) \\
\text{H}^0_T(M/K) \to \text{H}^0_T(L/K)
\end{array}
\]

\[
\begin{array}{c}
\text{H}^0_T(M/K) \to \text{H}^0_T(L/K) \\
\text{H}^2(M/K) \to \text{H}^2(L/K)
\end{array}
\]

$K^* / N_{M/K} M^* \to \mathbb{Z}$

$\text{cycl}_L$ is a cycle by $\text{tr}$
Computation via $H^0_T$

For each finite Galois extension $L/K$, we have $\mathcal{H}^2(L/K) = \text{cyc}$ of order $|L/K|$.
And now... statement of Tate's theorem

Let $G$ be a finite group. Let $M$ be a $G$-module.

Suppose $\forall H \leq G$ subgroups, $\text{Hom}(H, M) = 0$.

Then, there exist isomorphisms

$$\mu_i^*(G, \mathbb{Z}) \overset{\sim}{\longrightarrow} H_1^i(G, M)$$

which are determined by a choice of surjective

$$\text{Hom}(G, M)$$

Take $i = 2$:

$$\mu_2^*(G, \mathbb{Z}) \cong H_2^1(G, \mathbb{Z})$$

$$G^a_b = \text{Hom}(G, \mathbb{Z}) = K^*/\text{Ann}_{\mathbb{Z}} K^*$$
Proof of Tate's theorem: a key exact sequence

Let $\mathbb{G}$ be a group and $M$ a $\mathbb{G}$-module represented by a cyclic module $\phi: \mathbb{G} \to M$

Define an $\mathbb{G}$-module $M(\phi)$ (lifting module) as follows:

- $M(\phi)$ is a $\mathbb{G}$-module for which $\phi$ lifts uniquely.
- It has an exact sequence

$$0 \to M \to M(\phi) \to \mathbb{R}(\phi) \to \mathbb{R} \to 0$$

where

$$H^1_t(\mathbb{G}, \mathbb{Z}) \cong H^1_t(\mathbb{G}, \mathbb{Z}) \Rightarrow H^1_t(\mathbb{G}, M)$$
Definition of the splitting module

\[ M(\mathcal{D}) = M \otimes \bigoplus_{g} \mathbb{Z} \] 
\( g \in G \sim \{e\} \)

with \( \mu = \mu_{\alpha} \)

\[ x_{g} = x_{rg} - x_{g} + \phi(e_{g}, h_{g}) \]

where \( x_{e} = \phi(e_{e}, e) \).

Properties of \( \phi \) imply that this gives a \( G \)-action:
\[ \phi(g_{1}, g_{2}, g_{3}) = \phi(g_{1} g_{2}, g_{3}) \]
\[ \phi(1, g_{1}, g_{2}, g_{3}) - \phi(g_{1}, g_{2}, g_{3}) = \phi(g_{1} g_{2}, g_{3}) - \phi(1, g_{2}, g_{3}) = 0. \]

By construction, \( x_{e} \) is mapped to zero in \( H^{2}(e, M(\mathcal{D})) \):
\[ \phi(e, e) = x_{g}. \]
Towards acyclicity of the splitting module

\[ 0 \to M \to M(\alpha) \to I_\alpha \to 0 \]

with \( x_5 \to (y) - 1 \)

Let the long exact sequence:

\[ 0 \to H^1(\mathcal{M}) \to H^1(\mathcal{M} \cap \mathcal{P}) \to H^1(\mathcal{M}, \mathcal{P}) \to H^2(\mathcal{M}, \mathcal{P}) \to 0 \]

Be divided by \( \alpha \) in 0.

\[ \left[ \begin{array}{c}
-\gamma_j(1, \beta) \\
\gamma_j(1, \beta) \\
\gamma_1(1, \beta)
\end{array} \right] \]

\[ = 0 \]

\[ \text{fund H}_2 \text{H}_2 \text{H}_2 \text{H}_2 \]
Lemma: a shortcut to acyclicity (positive indices)

\[ \text{Let } G \text{ be a finite group } \]
\[ M = \langle b \rangle \text{ mod } 2 \quad \text{ for } i \leq s \text{ and } \]
\[ \suppose \ M'(1, M) = 0 \forall i \quad \text{ all } i \leq l, \]
\[ \text{Then } \mu_i(G, M) = 0 \forall i. \]

\[ \suppose \text{ for the moment that } \epsilon > 0 \text{ be small. } \]
\[ \text{Suppose in } \mathcal{H}, \text{ rich MRC s.t. } \| \epsilon \| \text{ be small. } \]
\[ \text{Given } \mu_i(G, M) = 0. \text{ In this hypothesis (i.e., } \epsilon = 0 \)
\[ \Rightarrow \mu_i(G, M) \quad \text{ by definition. } \]
\[ \Rightarrow \mu_i(G, M) \quad \text{ (6/17, } \text{ etc. } \mu_i(G, M) \Rightarrow 17, 1/17, M) \]
\[ \text{which then } G \text{ must satisfies hypotheses } \]
\[ \Rightarrow \text{ with } G \text{ cyclic (i.e., } \epsilon > 0 \text{). } \]
Lemma: a shortcut to acyclicity (negative indices)

For negative indices, the dimension of \( H^i \) is zero for \( i > 0 \).

\[
0 \to \mathbb{N} \to \prod_{i \leq 0} H^i \to 0
\]

\[
H^i(M) \cong H^i(W) \quad (m = 0, -1, \ldots)
\]
Lemma: a shortcut to acyclicity (nonsolvable case)

If $G$ not solvable, let $p$ be any prime $G_p = P 
solvable$. Subgroup $(solvable)$

Claim:

$H(G, M) \rightarrow H(G_p, M) \rightarrow H(G_p, M)$ injective on nonsolvable part

Proof: Let $S = C_G(b_p)$ be some subgroup $S$ of $G$.

So $S$ contains all of $H(G, M)$ which is closed for all $p$ (He says known to be torsion).