# From Tate's theorem to local class field theory

I have finally updated the PDF version of the CFT notes to catch up to the changes I have been making in the HTML version. However, going forward I will update the HTML regularly (typically before and/or after each lecture) but the PDF only sporadically. Reminder: the HTML version is posted at https://kskedlaya.org/cft/.

# Reminder: Tate's theorem

Let $G$ be a finite group

$M = G$-module

Suppose $\forall H \leq G$ subgroup

$H^1(H, M) = 0$

$H^2(H, M) =$ cyclic of order $\# H$

**Then** $\exists$ isomorphisms

$$H^i_T(G, \mathbb{Z}) \to H^{i+2}_T(G, M)$$

which depend only on choice of generator of $H^2(G, M)$.

# Application of Tate's theorem to local reciprocity

$L/K$ finite Galois extension of finite extensions of $\mathbb{Q}_p$

$G = \mathrm{Gal}(L/K)$ $\quad M = L^*$ $\quad H^1(H, M) = 0$ (Thm 90)

$\Leftarrow$ $\quad H^2(H, M) =$ cyclic of order $\#H$

(previous calculations)

$H^2(\bar{K}/K) \cong \mathbb{Q}/\mathbb{Z}$

$H_T^i(G, \mathbb{Z}) \cong H_T^{i+2}(G, L^*)$ (Tate thm)

take $i = -2$:

$G^{ab} \cong K^*/\mathrm{Norm}_{L/K} L^*$

$\longleftarrow$ the map to be induced by reciprocity.

Why do these fit together to give $\phi_K : K^* \to \mathrm{Gal}(K^{ab}/K)$?

# The norm limitation theorem

$M/K$ a finite Galois extension

$\cup \; L/K$, $\downarrow G$

Need this diagram:

$$
\begin{tikzcd}
K^*/\mathrm{Norm}_{M/K} M^* \arrow[r, "\Pi"] \arrow[d] & \mathrm{Gal}(M/K)^{ab} \arrow[d] \\
K^*/\mathrm{Norm}_{L/K} L^* \arrow[r, "\Pi"] & \mathrm{Gal}(L/K)^{ab}
\end{tikzcd}
$$

$\circlearrowleft$? $\quad G^{ab}$?

$\Updownarrow$

$$
\begin{tikzcd}
H^0_T(\mathrm{Gal}(M/K), M^*) \arrow[r] \arrow[d] & H^{-1}_T(\mathrm{Gal}(M/K), I_{\mathrm{Gal}(M/K)}) \arrow[d] \\
H^0_T(\mathrm{Gal}(L/K), L^*) \arrow[r] & H^{-1}_T(\mathrm{Gal}(L/K), I_{\mathrm{Gal}(L/K)})
\end{tikzcd}
$$

$\circlearrowleft$?

$1 \to M^* \to M^*(\phi_M) \to I_{\mathrm{Gal}(M/K)} \to 1$

$1 \to L^* \to L^*(\phi_L) \to I_{\mathrm{Gal}(L/K)} \to 1$

Take $\phi_L$ to represent a class whose inflation is $[G:H]$ times class of $\phi_M$.

# A special case of the local existence theorem

Need to show: every open, finite index subgp. of $K^*$ is $\mathrm{Norm}_{L/K}\, L^*$ for some finite ~~abelian~~ extension $L/K$

norm limitation theorem

Lemma:

Suppose $K \ni \zeta_\ell$ for some prime $\ell$ (where $\ell = p$ is OK)

Then $x \in K^*$ is an $\ell$-th power $\iff x \in \mathrm{Norm}_{L/K}\, L^*$ for every $\mathbb{Z}/\ell\mathbb{Z}$-extension $L/K$.

Pf $M$ = compositum of all $\mathbb{Z}/\ell\mathbb{Z}$-extensions. This is a finite extension of $K$.

So $\mathrm{Gal}(M/K) \cong (\mathbb{Z}/\ell\mathbb{Z})^n$ for some $n$.

Kummer theory $\Rightarrow$ $K^*/(K^*)^\ell$. $\to$ forced to have a map.

Local reciprocity: $K^*/\mathrm{Norm}_{M/K} M^* \cong (\mathbb{Z}/\ell\mathbb{Z})^n$

# A word on Hilbert symbols

Similarly if $\ell$ is not prime .
gives rise to Hilbert symbols

# Universal norms

Cor The intersection $\bigcap_{\substack{L/K \\ \text{finite Galois/abelian}}} \mathrm{Norm}_{L/K} L^* = \{1\}$.

Pf By previous slide, everything on the left is in $(K^*)^\ell$ $\forall \ell$ prime. [and all powers of $p$]

Check: this intersection is trivial!

(Hint: check intersection is in $\mathcal{O}_K^*$ then in 1-units)

# Proof of the local existence theorem

$U \subseteq K^*$ open finite index

need to show $\mathrm{Norm}_{L/K} L^* \subseteq U$

for some $L/K$ (not necessarily abelian).

take $L/K$ abelian, then $\frac{K^*}{\mathrm{Norm}_{L/K} L^*} \cong \mathrm{Gal}(L/K)$

Image of $U$ in $K^*/\mathcal{O}_K^* \xrightarrow{v} \mathbb{Z}$ is $m\mathbb{Z}$ for some $m$.

Insist that $L$ contain unramified ext of $K$ of degree $m$,

will ensure $\mathrm{Norm}_{L/K} L^*$ lands in $m\mathbb{Z}$

need to take ensure $(\mathrm{Norm}_{L/K} L^*) \cap \mathcal{O}_K^* \subseteq U \cap \mathcal{O}_K^*$.

$\mathcal{O}_K^*$ is compact, so this open subgroup is closed, hence compact.

By previous slide, these have trivial intersection, so

$\bigcap_L (\mathrm{Norm}_{L/K} L^* \cap \mathcal{O}_K^*) \setminus (U \cap \mathcal{O}_K^*)$ is empty $\Rightarrow$ finite intersection property one of these is empty.

# Proof of the local existence theorem

# An alternate approach to existence: Lubin-Tate theory

A more constructive approach (a la Kronecker–Weber)
uses formal group

Formal group over $\mathcal{O}_K$ is a power series
$F(X,Y) \in \mathcal{O}_K[[X,Y]]$ which satisfies
(with zero constant term)
$F(F(X,Y),Z) = F(X,F(Y,Z)) \in \mathcal{O}_K[[X,Y,Z]]$ etc

For $L/K$ finite, this gives well-defined group law on $\mathfrak{m}_L$: use torsion elements for such group laws
to get abelian extensions of $K$.

# Preview: abstract class field theory

Plan 13 for proving local reciprocity:
establish a similar result given:

- a field $K$
- a surjective hom $d : \underbrace{\mathrm{Gal}(\bar{K}/K)}_{=G} \to \hat{\mathbb{Z}}$
  (picks out "unramified" extensions)
- a discrete $G$-module $A$ (in our example $A = \bar{K}^*$)
- $v : \underbrace{A_K}_{=A^G} \to \hat{\mathbb{Z}}$ satisfying some conditions wrt $d$
  "abstract Henselian valuation"

**Et cetera** Class field axiom

for $L/K$ finite cyclic extension of finite extensions of $\mathbb{Q}_p$,

$$\hat{H}_T^i(\mathrm{Gal}(L/K), A_L) = \begin{cases} (L:K) & i=0 \\ 0 & i=1 \end{cases}$$

$A_L = A^{\mathrm{Gal}(\bar{K}/L)}$