Abstract class field theory

I've made quite a few updates to both the HTML and PDF versions of the CFT notes recently, mostly in sections we have not covered yet.

A remark about the existence theorem

K = finite ext of $\mathbb{Q}_p$

$D_K = \bigcap_{L/K \text{ finite}} \mathrm{Norm}_{L/K} L^*$ ← "universal norms"

<u>Lemma:</u> $D_K \subseteq \bigcap_{\ell \text{ prime}} (K^*)^\ell$ (could be improved using n instead of ℓ)

<u>exercise:</u> $\bigcap_n (K^*)^n = \{e\}$ (<u>does not</u> work with primes)

Claim: these together imply $D_K = \{e\}$.

<u>pf</u> first note $D_K \subseteq \mathcal{O}_K^*$, so $D_K = \bigcap_{L/K} \mathrm{Norm}_{L/K} \mathcal{O}_L^*$ <u>is compact</u>.

- $\mathrm{Norm}_{L/K} D_L = D_K$ (i.e. $\varprojlim_L D_L \twoheadrightarrow D_K$) ✓
- For $x \in D_K$, it has at most ℓ ℓ-th roots in K^*, and at least one must be in D_K.

A remark about the existence theorem

Why is $\mathrm{Norm}_{L/K} D_L \supseteq D_K$?

Given $x \in D_K$, for each L/K, $U_L := \mathrm{Norm}^{-1}_{L/K}(x) \subseteq O_L^\times$
$\subseteq \hat{O}_K$

is compact & nonempty.

Thm Every inverse limit of nonempty compact topological spaces is nonempty.

$\Uparrow$

Thm (Tikhonov) A product of compact topological spaces is compact.

(inverse limit is a closed subspace of product)..

Abstract ramification theory

Let k be a field. Want to relate $\mathrm{Gal}(L/K)^{ab}$, where $L/K/k$ finite extensions & L/K Galois, to "norm groups". Need three extra structures.

<u>1.</u> $d : \mathrm{Gal}(K^{sep}/K) \longrightarrow \hat{\mathbb{Z}}$

continuous, surjective

Think of fixed field of ker(d) as "K^{unr}".

in local CFT: $k = \mathbb{Q}_p$

$\hat{\mathbb{Z}} = \mathrm{Gal}(K^{unr}/K)$

The fundamental identity

L finite ext of k,

K/k finite

$L^{unr} = LK^{unr}$

$$d_K = \frac{1}{[\hat{\mathbb{Z}} : d(G_K)]}\, d : G_K \longrightarrow \hat{\mathbb{Z}}$$

abstract "ramification degree", "inertia degree"

$$e_{L/K} = [I_K : I_L] \qquad f_{L/K} = [d(G_K) : d(G_L)]$$

$I_K = \ker(d_K)$

$$\underline{e_{L/K} f_{L/K}} = [G_K : G_L] = [L : K].$$

Abstract unit groups

$G = \mathrm{Gal}(k^{sep}/k)$ (profinite)

k = local field

2: Let A be a discrete G-module (written multiplicatively)

analogue of $(k^{sep})^* \circlearrowleft \mathrm{Gal}(k^{sep}/k)$ with fixed $= k^*$.

$A_L = A^{\mathrm{Gal}(k^{sep}/L)}$

L/K finite: $\mathrm{Norm}_{L/K} : A_L \to A_K$

(not necessarily Galois)

L/K infinite, $\mathrm{Norm}_{L/K} A_L = \bigcap_{\substack{M/K \text{ finite} \\ \subseteq L/K}} \mathrm{Norm}_{M/K} A_M$

The class field axiom

Assume for every cyclic extension L/K of finite extensions of k.

$$\# H_T^i(\mathrm{Gal}(L/K), A_L) = \begin{cases} [L:K) & i = 0 \\ 1 & i = -1. \end{cases}$$

(Note: not enough to take $K = k$)

Abstract henselian valuations

3. Need $v : A_k \to \hat{\mathbb{Z}}$ s.t.

1) $\Sigma = \mathrm{im}(v)$ contains $\mathbb{Z}$ and $\Sigma / n\Sigma = \mathbb{Z}/n\mathbb{Z}$ $\forall n$
2) for every finite ext K/k, $\underbrace{v(\mathrm{Norm}_{K/k} A_K) = f_{K/k}\,\Sigma}$;

Define $v_K = \dfrac{1}{f_{K/k}} v \circ \mathrm{Norm}_{K/k}$.

Cohomology of the abstract units

Cohomology of the abstract units

$k \underset{\text{finite}}{\subset} K \subset k^{sep}$

$U_K := \ker v_K : A_K \to \hat{\mathbb{Z}}$ (analogue of $\mathcal{O}_K^\times$)

Prop For L/K an "unramified" extension of finite ext of k ($e_{L/K} = 1$)

$\Rightarrow H^i_T(\underset{\text{cyclic}}{\overset{=G}{Gal}}(L/K), U_L) = 1$ $\forall i$ (i.e. for $i = 0$, $i = -1$)

and $H^i_T(Gal(L/K), A_L) = \begin{cases} \underline{(L:K)} \text{ cyclic} & i = 0 \\ \underline{1} & i = -1 \end{cases}$

$$\cancel{H^{-1}_T(v(A_L))} \hookrightarrow H^0_T(U_L) \to \underset{(L:K)}{H^0_T(A_L)} \to \underset{=(L:K)}{H^0_T(v(A_L))} \to H^1(U_L) \to \cancel{H^1(A_L)}$$

check this is

~ [illegible] function trivial

$$1 \to U_L \to A_L \to v(A_L) \to 1$$

First definition of the abstract reciprocity map

L/K finite ext.

$$H = \{ g \in \mathrm{Gal}(L^{unr}/K) : d_K(g) > 0 \}.$$

$$r' : H \to A_K / N_{L/K} A_L$$

w/ $g \in H$, M = fixed field of g

$$r'(g) = \mathrm{Norm}_{M/K}(\pi_M)$$

π_M = uniformizer (element of A_K maps to 1 in $A_K/U_K \cong \mathbb{Z}$)

does not depend on π_M (by previous lemma)

Second definition of the abstract reciprocity map

Multiplicativity of the reciprocity map

Alternate formula: Choose $\phi \in G$

$$d_K(\phi) = 1$$

$$d_K(g) = n$$

$\forall x \in A_M$

$$\mathrm{Norm}_{M/K}(x) = \mathrm{Norm}_{\underbrace{L^{unr}/K^{unr}}}(x\, x^{\phi} \cdots x^{\phi^{n-1}})$$

Statement of the abstract reciprocity law

Goal: the map $r': M \to A_K / \mathrm{Norm}_{L/K} A_L$ factors through an isomorphism.

$$\begin{tikzcd}
\mathrm{Gal}(L/K)^{ab} \arrow[r, "r", "\sim"'] & A_K/\mathrm{Norm}_{L/K} A_L \\
\mathrm{Gal}(L^{ntr}/K) \arrow[u] \arrow[ur, "r'"'] &
\end{tikzcd}$$