The reciprocity law in abstract class field theory

PS 14 has been posted. It is due Thursday, February 18.

Warning: I am planning to reorder a couple of sections in the notes. In particular, the section we will cover after this (ramification filtrations) will end up *before* abstract CFT, which will be moved into a new chapter.

Reminder: the setup of abstract CFT

k = field, $\bar{k}$ = any separable algebraic extension

$$G = \mathrm{Gal}(\bar{k}/k)$$

A = G-module, $A_K = A^{\mathrm{Gal}(\bar{k}/K)}$

$d : \mathrm{Gal}(\bar{k}/k) \to \hat{\mathbb{Z}}$ continuous surjection

$\rightsquigarrow e_{L/K}, f_{L/K}$

$v : A_k \to \hat{\mathbb{Z}}$

$$\boxed{v(\mathrm{Norm}_{K/k} A_K) = f_{K/k}\, v(A_k)}$$

K not necessarily Galois

e.g. $k = \mathbb{Q}_p$, $A = \bar{\mathbb{Q}}_p^*$

Class field axiom

for L/K cyclic

$$\boxed{\# \hat{H}^i_T(\mathrm{Gal}(L/K), A_L) = \begin{cases} (L:K) & i=0 \\ 1 & i=1 \end{cases}}$$

K, L finite extensions of k within $\bar{k}$

The reciprocity map: first formula

L/K finite Galois

Let H = semigroup of $g \in \mathrm{Gal}(\underbrace{L^{unr}}_{}/K)$ s.t. $d_K(g)$ is positive integer

$L \cdot \underbrace{\mathrm{Fix}(\ker(d))}_{K^{unr}}$

define

$r : H \to A_K / \mathrm{Norm}_{L/K} A_L$

let $M = \mathrm{Fix}(g)$ $\quad e(M/K) = e(M \cap L/K) \quad f(M/K) = d_K(g)$

define $r(g)$ = class of $\mathrm{Norm}_{M/K} \pi_M$ $\quad \pi_M$ = uniformizer of M

(i.e. $v_M(\pi_M) = 1$)

Note: replacing π_M changes this by an element of

$U_K \cap \mathrm{Norm}_{L^{unr}/K} A_{L^{unr}} = \mathrm{Norm}_{L/K} A_L$ when L/K unram

follows from $H^i_T(\mathrm{Gal}(L/K), U_L) = 0$

The reciprocity map: second formula

$n = d_K(g)$

Lemma Pick $\phi \in M$, $d_K(\phi) = 1$

$\mathrm{Norm}_{M/K}(x) = \mathrm{Norm}_{L^{unr}/K^{unr}}(x x^{\phi} \cdots x^{\phi^{n-1}})$

(e.g. take $x = \pi_M$)

Pf $U = M \cap K^{unr}$ so $\mathrm{Norm}_{M/K} = \mathrm{Norm}_{U/K} \circ \mathrm{Norm}_{M/U}$

U/K unram. of degree n, so for $y \in A_U$, $\mathrm{Norm}_{U/K}(y) = y y^{\phi} \cdots y^{\phi^{n-1}}$

Meanwhile, $\mathrm{Norm}_{M/U}$ on A_M is restriction of $\mathrm{Norm}_{L^{unr}/K^{unr}}$.

Multiplicativity of the reciprocity map

Lemma: $r' : H \to A_K / \mathrm{Norm}_{L/K} A_L$ is a homomorphism of semigroups

Pf $g_1 g_2 = g_3$ in H

$$\underbrace{r'(g_1)}_{\rho_1} \cdot \underbrace{r'(g_2)}_{\rho_2} \overset{?}{=} \underbrace{r'(g_3)}_{\rho_3}$$

First, valuations match.

$$v_K(\rho_i) = \underbrace{f(M_i/K)\, v_{M_i}(\pi_i)}_{d_K(g_i)}, \quad \underbrace{v_{M_i}(\pi_i)}_{1}$$

Want to show $\rho_1\rho_2/\rho_3 \in \mathrm{Norm}_{L/K} A_L$

enough it's in U_K.

Previous slide: $\rho_i := \mathrm{Norm}_{L^{unr}/K^{unr}}(\sigma_i)$ $\quad \sigma_i = \pi_i \pi_i^{\phi} \cdots \pi_i^{\phi^{d_K(g_i)-1}}$

$$\underbrace{\rho_1\rho_2/\rho_3}_{} = \mathrm{Norm}_{\cancel{L^{unr}}/K^{unr}}\left(\frac{\sigma_1\sigma_2}{\sigma_3}\right)$$ for some N/L finite unram.

$$U_K \cap \mathrm{Norm}_{N/N\cap K^{unr}} U_N \subseteq \mathrm{Norm}_{N/K} U_K$$

Cohomology of the abstract units

Lemma: Let $M/L/K$ be finite extensions (finite with M/K Galois and L/K unramified). weak

Then $U_K \cap \mathrm{Norm}_{M/L} U_M = \mathrm{Norm}_{M/K} U_K$.

(if $M=L$, follows from class field axiom)

Pf: $H^0_T(\mathrm{Gal}(L/K), U_L)=0 \Rightarrow H^0_T(\mathrm{Gal}(M/K), U_M) \overset{\mathrm{Res}}{\longleftrightarrow} H^0_T(\mathrm{Gal}(M/L), U_M)$

If this were H^1 instead of H^0_T, this would be inflation-restriction

Lemma: $H \trianglelefteq G$, M is a G-mod, $\longleftrightarrow$ N G-module

$H^0_T(G/H, M^H)=0 \Rightarrow H^0_T(G,M) \xrightarrow{\mathrm{res}} H^0_T(H,M)$ $\quad H^1(G/H, M^H)=0 \Rightarrow \ldots$

Reciprocity in the unramified case

$r_{L/K}$: $\mathrm{Gal}(L/K) \longrightarrow A_K / N_{L/K} A_L$

Lemma: if L/K is unramified, then $r_{L/K}$ is an isom.

and "Frobenius" $\longrightarrow \pi_K$

(read definition)

Reciprocity in the cyclic ramified case

$$\begin{tikzcd} & N \arrow[dl, dash] \arrow[dr, dash] & \\ M \arrow[dr, dash] & & L \arrow[dl, dash] \\ & K & \end{tikzcd}$$

Lemma L/K cyclic totally ramified

then $r_{L/K} : \mathrm{Gal}(L/K) \to A_K / \mathrm{Norm}_{L/K} A_L$ is an isomorphism.

Pf note same order by class field axiom

$\Rightarrow$ only need to check injective.

$\mathrm{Gal}(L^{unr}/K) = \mathrm{Gal}(L/K) \times \mathrm{Gal}(K^{unr}/K)$

$\langle \sigma \rangle$

$= \mathrm{Gal}(L^{unr}/K)$

$\langle \phi \rangle \quad d_K(\phi) = 1$

$\tau = \sigma\phi$ has $d_K(\tau) = 1$ $\quad M =$ fixed field of τ

$N = LM$

Reciprocity in the cyclic ramified case

$n = (L:K)$. Let j be min positive int s.t. $r(g^j) \in \mathrm{Norm}_{L/K} A_L$

$\stackrel{?}{\Rightarrow} n \mid j$.

Calculate: $r(g) = \mathrm{Norm}_{L^{unr}/K^{unr}}(\pi_M / \pi_L)$

$u = (\pi_L / \pi_M)^j \quad \mathrm{Norm}_{L^{unr}/K^{unr}}(u) = \mathrm{Norm}_{L/K}(v)$
$v \in U_L$

write $\frac{u}{v} \in \frac{a^g}{a}$ by 1st class field axiom.

... $j \mid n$.

Reciprocity in the general case

$\mathrm{Gal}(L/K)^{ab} \twoheadrightarrow A_K/\mathrm{Norm}_{L/K} A_L$

$\mathrm{Gal}(M/K) \twoheadrightarrow A_K/\mathrm{Norm}_{M/K} A_M$

In general case

- for L/K abelian, induct down to cyclic case to get

$r_{L/K} : \mathrm{Gal}(L/K)^{ab} \longrightarrow A_K/\mathrm{Norm}_{L/K} A_L$

- get same in solvable case by compatibility

- handle nonsolvable case via Sylow subgroups