The reciprocity law in abstract class field theory

PS 14 has been posted. It is due Thursday, February 18.

Warning: I am planning to reorder a couple of sections in the notes. In particular, the section we will cover after this (ramification filtrations) will end up before abstract CFT, which will be moved into a new chapter.
Reminder: the setup of abstract CFT

\[ K = \text{field} \quad \overline{K} = \text{any } \frac{1}{16} \text{br. c. extension} \]

\[ \mathcal{C} = \text{Gal} (K/k) \]

\[ A = \text{modulo } \Lambda \text{Le} \quad A_K = A_{\text{Gal}(\overline{K}/k)} \]

\[ \phi: \text{Gal}(\overline{K}/k) \rightarrow \mathbb{Z} \text{ with unique } \phi \]

\[ \rightarrow E_{\overline{K}/K}, \text{ flux } \]

\[ V: A_K \rightarrow \mathbb{Z} \]

\[ \hat{V}(\text{Norm}_{K/k} A_K) = f_{K/k} \quad \hat{V}(A_K) \quad \text{not necessary on 0} \]

\[ \sum_{i=0}^{\infty} f_l (\mathcal{C}_{(l, k)}, A_{l}) = 0 \quad i = 0 \quad \sum_{i=1}^{\infty} f_l (\mathcal{C}_{(l, k)}, A_{l}) = 1\]

\[ K, L \text{ finite extensions of } K \text{ with } \mathcal{C} \]
The reciprocity map: first formula

Let \( M = \text{semisimp of } g \in \mathcal{G}L(\mathbb{C}^{n-1}/K) \) s.t. \( \delta(g) \) is \( L. \frac{1}{2} \text{F}(\text{reg}(\mathcal{H})) \) c.p. \( v_{\mathcal{H}} \).

Let \( r : 1 \rightarrow \mathbb{A}^+ \to \text{Norm}(1)_K \) be

\( \text{Let } \mathcal{M} = F_{\infty}(g) \)

\( \text{Note: replacing } \tilde{\mathcal{H}} \text{ changes this by a } \mathcal{H} \text{-isomorphism of } \mathbb{A}_L \to \mathbb{A}_L \)
The reciprocity map: second formula

\[ \Lambda_K(\phi) = \frac{1}{\phi^n - 1} \]

Given \( \phi \in \mathbb{G}_M \),

\[ \text{Norm}_{\pi/K}(x) = \text{Norm}_{\pi/K}(x \times \phi \times \ldots \times \phi^{n-1}) \]

(i.e., \( x = \pi m \))

Let \( U = \mathbb{M}_K \cup \mathbb{U}_K \cup \mathbb{M}_M \cup \mathbb{U}_M \),

\[ \text{Norm}_{\pi/K} \times \text{Norm}_{\pi/M} = \text{Norm}_{\pi/K} \circ \text{Norm}_{\pi/M} \]

\( U/K \) unram. of degree \( n \), \( SU \) for \( y \in A_K \), \( \text{Norm}_{\pi/K}(y) = y \)

Meanwhile, \( \text{Norm}_{\pi/M} \) on \( A_M \) is restriction of \( \text{Norm}_{\pi/K} \).

\[ \text{Norm}_{\pi/K} \cup \text{Norm}_{\pi/M} \]
Multiplicativity of the reciprocity map

Let \( r : H \to \operatorname{Aut}(\mathcal{V}_K) \). We have

\[ r_1(s_1) \cdot r_2(s_2) = r_3(s_3), \]

with \( s_i \in S_i \). Let \( \mathcal{V}_K \) be a representation.

First, we check that

\[ \nu_K(s_i) : f(M_i \mid K) \mu_{M_i}(s_i), \]

but to show \( s_1 s_2 / s_3 \in \operatorname{Num}_{\mathcal{V}_K} A_L \) for all \( i \), it's in \( \mathcal{V}_K \).

Define \( s_i = \nu_{M_i} L_{\mathcal{V}_K} (s_i) \), where \( s_i \in S_i \).

So, \( s_i \) is the image of \( s_i \) under \( \nu_{M_i} L_{\mathcal{V}_K} \) for some \( N_i \) in the image.

Thus, \( \mathcal{V}_K \) is a representation.

For some \( N_i \) in the image, we have

\[ \mu_{M_i}(s_i) \]
Cohomology of the abstract units

Let $M/L/K$ be finite extensions, with $M/K$ abelian and $L/K$ unramified. Then:

The $U_K \cap \text{Norm}_{M/L} U_L = \text{Norm}_{M/K} U_K$.

(If $M = L$, follows from class field theory.)

\[ H^1_{\text{fl}}(\mathbb{G}_m, U_L) = 0 \quad \Rightarrow \quad H^2_{\text{fl}}(\mathbb{G}_m, U_K) \cong H^0(L(1), \mathbb{Z}) \]

(If $h_M$ were $h_L$ instead of $1$, this would be inflation-restriction.

\[ H^0(\mathbb{G}_m, M) \cong N \otimes M \]

\[ H^1_{\text{fl}}(L(1), M) = 0 \Rightarrow H^1_{\text{fl}}(L(1), N) \rightarrow H^1_{\text{fl}}(L(1), M) \rightarrow H^1_{\text{fl}}(L(1), N(1)) = \cdots \]
Reciprocity in the unramified case

\[ \mathbb{E}_K \text{ Gal}(L/K) \to \mathbb{A}_K/\mathbb{G}_m \text{ at } \mathfrak{p} \]

Lemma: If \( L/K \) is abelian, then the \( \mathbb{A}_K \) is torsion. 

and "torsion" \( \to \mathbb{A}_K \)

( real division )
Reciprocity in the cyclic ramified case

Let $L/K$ be a cyclic Galois totally ramified extension.

Then $\text{Gal}(L/K) \cong \mathbb{Z}/m\mathbb{Z}$, where $m$ is the ramification index.

If $\phi$ is a norm in $L/K$, let $\phi$ be the same order as $\phi$ held earlier.

Then $\text{Gal}(L^{ur}/K) = \text{Gal}(L/K) \times \text{Gal}(K^{ur}/K).$

Let $\tau = \phi \circ \text{res} \circ \phi^{-1}$.

$M = \text{fixed field of } \tau$ and $N = L_M$. 

$N = \tau \sigma = L_M$.
Reciprocity in the cyclic ramified case

\[ n - \frac{C_{\ell}}{w_{2}} \leq \text{min. pos. s.t. } \sqrt{\nu} (q) \in \mathbb{N}_{\nu} \cup \mathbb{A} \]

\[ \Rightarrow n' \mid j. \]

Calculate: \[ (q) = \text{Norm}_{\mathbb{L}/\mathbb{K}}(\nu_{1} \nu_{2} \nu_{3}) \]

\[ \nu = (\nu_{1} \nu_{2} \nu_{3}) \quad \text{Norm}_{\mathbb{L}/\mathbb{K}}(\nu) = \text{Norm}_{\mathbb{L}/\mathbb{K}}(v) \quad \text{we write} \]

\[ \nu = \frac{c_{2}}{c_{1}} \quad \text{by 1st class field axiom.} \]

\[ \ldots \quad j | n. \]
Reciprocity in the general case

In general case

- for like abelian, induct down, cyclic case to tet

\[ \text{L} : \text{Gal}(L/K) \rightarrow \text{Ab} / \text{Norm} \text{L/K} \]

- let some is soluble case be compatible 8.1

- handle nonsoluble case as in local theory