Ramification filtrations and local class field theory

No lecture or office hours Monday, February 15 (university holiday).

As promised, the CFT notes have been reorganized: abstract class field theory is now its own chapter (Chapter 5), while the section on ramification filtrations (newly added for this course) remains in Chapter 4.

Finishing up from last time: the abstract reciprocity law

k = field, $d : \mathrm{Gal}(\bar{k}/k) \twoheadrightarrow \hat{\mathbb{Z}}$ ← defines "unramified" extensions

$A \circlearrowleft G = \mathrm{Gal}(\bar{k}/k)$

$v : A_k \to \hat{\mathbb{Z}}$ henselian valuation

Last time: for L/K finite Galois extension defined a homomorphism

$$\begin{tikzcd}
\mathrm{Gal}(L/K) \arrow[r] \arrow[d] & A_K / \mathrm{Norm}_{L/K} A_L \\
r : \mathrm{Gal}(L/K)^{ab} \arrow[ur, "r"'] &
\end{tikzcd}$$

Claim: $r : \mathrm{Gal}(L/K)^{ab} \to A_K / \mathrm{Norm}_{L/K} A_L$ is an isomorphism

<u>Lemma</u>: This is true if L/K is unramified (easy calculation)
or L/K is cyclic totally ramified (harder calculation)

Finishing up from last time: the abstract reciprocity law

$L/M/K$ Galois

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(L/M) \arrow[r] \arrow[d, "r_{L/M}"] & \mathrm{Gal}(L/K) \arrow[r] \arrow[d, "r_{L/K}"] & \mathrm{Gal}(M/K) \arrow[r] \arrow[d, "r_{M/K}"] & 1 \\
& A_M/\mathrm{Norm}_{L/M} A_L \arrow[r, "\mathrm{Norm}"] & A_K/\mathrm{Norm}_{L/K} A_L \arrow[r] & A_K/\mathrm{Norm}_{M/K} A_M \arrow[r] & 1
\end{tikzcd}$$

- $r_{L/K}$ surjective when L/K is solvable.
- $r_{L/K}$ isomorphism when L/K is cyclic (class field axiom).
- $r_{L/K}$ isomorphism when L/K is abelian (write Galois as product of cyclics)
- $r_{L/K}: \mathrm{Gal}(L/K)^{ab} \to A_K/\mathrm{Norm}_{L/K} A_L$ is injective in general
  (compare L/K with $M/K = L^{ab}$
- $r_{L/K}$ surjective in general: take $M/K = \mathrm{Fix}(\text{Sylow } p\text{-sub})$
  (not Galois!!) $[M:K]$ not div by p.

Consequences: norm limitation etc.

Cor L/K any finite extension with M/K maximal subextension, then $\mathrm{Norm}_{L/K} A_L = \mathrm{Norm}_{M/K} A_M$.

Cor L_1/K, L_2/K abelian,

$\mathrm{Norm}_{L_1/K} A_{L_1} = \mathrm{Norm}_{L_2/K} A_{L_2} \Rightarrow L_1 = L_2$.

(Pf. $L = L_1 L_2/K$, then $\mathrm{Gal}(L/L_1) = \mathrm{Gal}(L/L_2)$.)

To prove existence theorem: need to figure out which subgroups of A_K contain some $\mathrm{Norm}_{L/K} A_L$.

And now for something completely different...

But see "A look ahead" in the notes for a preview of how we will apply abstract CFT.

Ramification in the lower numbering

K = finite ext of $\mathbb{Q}_p$

<u>Claim:</u> Local CFT defines an isomorphism

$$\mathrm{Gal}(K^{ab}/K) \cong \widehat{K^*} \cong O_K^* \times \pi_K^{\hat{\mathbb{Z}}}$$

comes w/ a filtration by subgroups

$$U_K^i = \{x \in O_K^* : v_K(x-1) \geq i\}$$

However, we already have a filtration on $\mathrm{Gal}(L/K) = G$ for any finite Galois extension L/K:

$$G_i = \{g \in G : g \text{ acts trivially on } O_L/\pi_L^{i+1}\}.$$

<u>Q:</u> Are these constructions related when G is abelian?

The Herbrand functions

$G = \mathrm{Gal}(L/K)$ finite

$\varphi_{L/K} : [-1, \infty) \to [-1, \infty)$ $\varphi_{L/K}(u) = \int_0^u \frac{dt}{[G_0 : G_t]}$

continuous, piecewise linear, increasing

(concave, unbounded $\Rightarrow$) bijective $G_t := G_{\lceil t \rceil}$

$\psi_{L/K} = \varphi_{L/K}^{-1} : [-1, \infty) \to [-1, \infty)$

warning: $\varphi_{L/K}$, $\psi_{L/K}$ do not act on $\mathbb{Z} \cap (-1, \infty)$! only on $\mathbb{Q} \cap (-1, \infty)$

$L/K'/K$ L/K', K'/K Galois

$$\boxed{\begin{aligned} \varphi_{L/K} &= \varphi_{K'/K} \circ \varphi_{L/K'} \\ \psi_{L/K} &= \psi_{L/K'} \circ \psi_{K'/K} \end{aligned}}$$

Ramification in the upper numbering

$G = \mathrm{Gal}(L/K)$

$$G^i = G_{\psi_{L/K}(i)} \iff G_i = G^{\varphi_{L/K}(i)}$$

The <u>breaks</u> in the upper numbering are in general only in $\mathbb{Q}$, not $\mathbb{R}$

6.1: Theorem (Herbrand)

For $H \trianglelefteq G$ normal $H = \mathrm{Gal}(L/K')$

$$(G/H)_i = G_{\psi_{L/K'}(i)} H/H \implies (G/H)^i = G^i H/H$$

(image of $\uparrow$ in G/H)

$= \mathrm{Gal}(L/K')$

$$\boxed{\begin{array}{l} \text{For } H \subseteq G \\ H_i = G_i \cap H \end{array}}$$

Compatibility with quotients

G, $H \trianglelefteq G$ normal $H = \mathrm{Gal}(L/K')$

$(G/H)_i = G_{\varphi_{L/K'}(i)} H/H \Rightarrow (G/H)^i = G^i H/H$

(image of $\uparrow$ in G/H)

$\Downarrow$

The upper numbering filtration extends to Galois groups of infinite algebraic extensions of K, such as $\mathrm{Gal}(\bar{K}/K)$ or $\mathrm{Gal}(K^{ab}/K)$

(i.e. G^i for one of these is inverse limit of G^i in the finite quotients.)

The Hasse-Arf theorem

In general, the breaks of ~~ramification~~ upper numbering filtration are not integers. (see HW)

<u>Thm</u> (Hasse-Arf) if L/K is (finite) abelian extension of K = finite ext of $\mathbb{Q}_p$

then the breaks in the upper # filtration on $\mathrm{Gal}(L/K)$ are integers!

Reciprocity and the ramification filtration

Thm L/K finite abelian extension of finite exts of $\mathbb{Q}_p$

Let $r_{L/K}: K^*/\mathrm{Norm}_{L/K} L^* \xrightarrow{\sim} G = \mathrm{Gal}(L/K)$ be the reciprocity isomorphism

Then inverse image of G^i in

$$O_K^* \to K^* \to K^*/\mathrm{Norm}_{L/K} L^* \to G$$

is U_K^i.

Example: the cyclotomic case

e.g. $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$

has breaks at $1, 2, \ldots, n$

in the upper numbering.

Breaks in the ramification filtration

Conductors of Artin representations

Hasse-Arf $\Longrightarrow$ for an Artin representation

$$\rho : G_K \xrightarrow[\text{Artin}]{} GL_n(\mathbb{C})$$

Can define conductor and it will always be a ~~non~~ nonnegative integer.