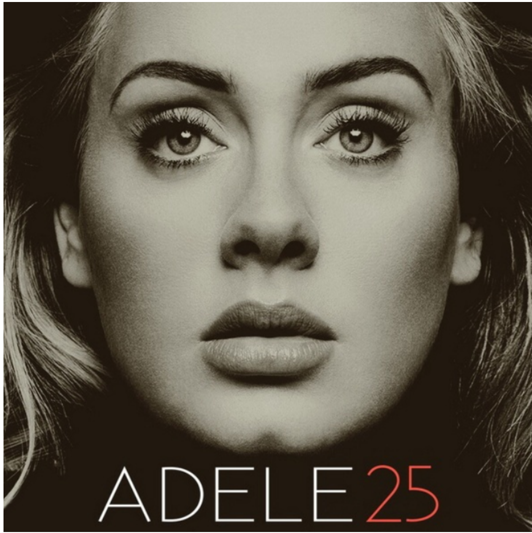
Adèles

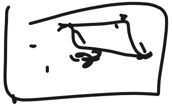
HW 15 has been posted.

Note: the former section 6.1 in my CFT notes has been split in two. This lecture corresponds to the new 6.1; the next lecture will correspond to the new 6.2. (I've also updated the pointers on the web site.)



The Minkowski space of a number field

Let K be a number field of degree n



$$K_{\mathbb{C}} = K \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\tau} \mathbb{C}$$

where τ runs over the n embeddings $K \hookrightarrow \mathbb{C}$

K F = conjugation

$$K_{\mathbb{R}} = K_{\mathbb{C}}^{F} = K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{s_1} \times \mathbb{C}^{s_2} \qquad \dim_{\mathbb{R}} K_{\mathbb{R}} = n$$

$K_{\mathbb{C}}$ carries standard Hermitian inner product $\Rightarrow$ restricts to a pos def inner product on $K_{\mathbb{R}}$

Any fractional ideal of K embeds into $K_{\mathbb{R}}$ as discrete cocompact subgroup (called a lattice in $K_{\mathbb{R}}$)

The profinite completion of a number ring

Recall: $\hat{\mathbb{Z}} = \varprojlim_{n} \mathbb{Z}/n\mathbb{Z} \underset{(\text{CRT})}{\cong} \prod_{p} \mathbb{Z}_p$ $\quad \mathbb{Z}_p = \varprojlim_{m} \mathbb{Z}/p^m\mathbb{Z}$

$= \varprojlim_{n} \mathbb{Z}/n!\mathbb{Z}$

Lemma K = # field

similarly $\widehat{\mathcal{O}_K} = \prod_{\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}$ where $\mathfrak{p}$ runs over (non zero) prime ideals of $\mathcal{O}_K$

$\qquad\quad \| \qquad\qquad\qquad$ compact topological spaces (Tikhonov's theorem)

note: same completion as ~~$\mathbb{Z}$-mod~~ ring $\quad \varprojlim_{n} \mathcal{O}_K/n\mathcal{O}_K$

The adèle ring of Q: finite part

(compare: $\frac{\mathbb{Q}}{\mathbb{Z}} = \bigcup_n \frac{1}{n}\mathbb{Z}$)

$\mathbb{A}_{\mathbb{Q}}^{fin} =$ • $\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$

• $\bigcup_n \frac{1}{n}\widehat{\mathbb{Z}}$ ← locally compact topological ring

• as ~~subset~~ proper subset of $\prod \mathbb{Q}_p$ (e.g. does not include $(1/2, 1/3, 1/5, \dots)$)

↓

restricted direct product of $(\mathbb{Q}_p, \mathbb{Z}_p)$

Restricted direct products

I = index set. For each $i \in I$, let (G_i, H_i) be a pair where $H_i \leq G_i$.

The restricted direct product of (G_i, H_i) is the set of tuples $g = (g_i)_{i \in I} \in \prod_i G_i$ s.t. $g_i \in H_i$ for all but finitely many indices i.

i.e. union of G_S over all finite subsets S of I

where $G_S = \prod_{i \in S} G_i \times \prod_{i \notin S} H_i \subseteq \prod_i G_i$

Restricted direct products

This also works in the following categories

- groups (H_i subgroup of G_i)
- rings (H_i subring)
- topological spaces
- locally compact topological spaces (H_i is a compact subspace)
- locally compact topological groups, topological rings

$\mathbb{A}^{fin}_{\mathbb{Q}}$

The adèle ring of Q

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_{\mathbb{Q}}^{fin} = \mathbb{R} \times \bigcup_n \tfrac{1}{n}\widehat{\mathbb{Z}}$$

locally compact topological ring

ring of adèles

$$\mathbb{Q} \hookrightarrow \mathbb{R} \times \mathbb{A}_{\mathbb{Q}}^{fin} \subset \mathbb{R} \times \prod_p \mathbb{Q}_p$$

diagonal embedding

(image = principal adèles)

$\mathbb{A}_{\mathbb{Q}}$ is restricted direct product of

$$(\mathbb{R}, \{0\}), (\mathbb{Q}_2, \mathbb{Z}_2), (\mathbb{Q}_3, \mathbb{Z}_3), \ldots$$

(note: restricted direct product of (G_i, H_i) is unchanged if you change finitely many H_i)

Q is discrete and cocompact

$\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$

$$U = \{x \in \mathbb{R} : |x|_\infty < 1\} \times \prod_p \underbrace{\{x \in \mathbb{Q}_p : |x|_p \le 1\}}_{=\mathbb{Z}_p}$$

$0 \in U$ in fact $U \cap \mathbb{Q} = \{0\}$ by product formula

if $x \in \mathbb{Q}^*$, then $|x|_\infty |x|_2 |x|_3 \cdots = 1$

the set $\underbrace{[0,1) \times \prod_p \mathbb{Z}_p}_{\text{compact}}$ surjects onto $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ (exercise)

hence $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is Hausdorff & compact.

The adèle ring of a number field

K = # field.

$$\mathbb{A}_K^{fin} = \widehat{\mathcal{O}_K} \otimes_{\mathcal{O}_K} K = \bigcup_n \frac{1}{n}\widehat{\mathcal{O}_K}$$

= restricted direct product of $(K_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})_{\mathfrak{p}}$ are

$$\mathbb{A}_K = K_\mathbb{R} \times \mathbb{A}_K^{fin}$$

= restricted direct product of $\left(K_v, \begin{cases} \{0\} & (v \text{ infinite}) \\ \mathcal{O}_{K_v} & (v \text{ finite}) \end{cases}\right)$

v are all places v of K

$K \hookrightarrow \mathbb{A}_K$ diagonal embedding.

elements of image are called principal adèles

The product formula

Normalize abs values on K_v:

- v real, usual abs value
- v complex, square of usual absolute value
  (warning: does not satisfy the $\triangle \leq$)
- v p-adic: normalize so $|p|_v = p^{-1}$

Thm: for $x \in K^*$, $\prod_v |x|_v = 1$.

(for places of $\mathbb{Q}$) $= \prod_w |\mathrm{Norm}_{K/\mathbb{Q}}\, x|_w = 1$

K is discrete in A_K

By product formula,

K is discrete in $\mathbb{A}_K$:

$U = \prod \{ x \in K_v : |x| < 1 \}$ ✓ infinite

✓ finite

OK ✓

is open in $\mathbb{A}_K$ & $U \cap K = \{0\}$.

K is cocompact (and the Chinese remainder theorem)

S = finite set of places of K

A_S = subring of $A_K \ni x$ s.t. $\forall$ finite $v \notin S$, $x_v \in O_{K_v}$.

"S-units of A_K"

prop For any S, $K + A_S = A_K$. ← "adelic CRT"

Cor A_K/K is compact.

Pf write down a compact subset U of $K_{\mathbb{R}}$ that covers $K_{\mathbb{R}}/O_K$. Then $U \times \widehat{O}_K$ is compact and covers A_K.

Preview: idèles and the idèle class group

Next time: study ~~unit group~~ I_K of $\mathbb{A}_K$.

(much smaller than $\mathbb{A}_K - \{0\}$!)

This will contain K^*

and I_K / K^* will be closely related

to $Cl(K)$.