Idèles and class groups

Reminder: the adèle ring of a number field

$K = \#$ field

$\mathbb{A}_K$ = restricted direct product over places v of K

of $\begin{cases} (K_v, \{0\}) & v \text{ infinite} \\ (K_v, \mathcal{O}_{K_v}) & v \text{ finite} \end{cases}$

i.e. $x \in \mathbb{A}_K$ means $x \in \prod_v K_v$

and $x_v \in \mathcal{O}_{K_v}$ for all but finitely many places v — finite v

Topologised as a locally compact topological ring

$\mathbb{A}_K = \bigcup_S \mathbb{A}_{K,S}$ — S finite set of places

$\mathbb{A}_{K,S} = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_{K_v}$

The idèle group of a number field

Group $I_K = \mathbb{A}_K^\times$ (group of units)

i.e. $x \in I_K$ if $x \in \prod_v K_v^*$ and $x^{-1} \in \mathbb{A}_K$

$x_v \in \mathcal{O}_{K_v}^*$ for all but finitely many finite v

i.e. I_K is a restricted direct product over v

of $\begin{cases} (K_v^*, \{1\}) & v \text{ infinite} \\ (K_v^*, \mathcal{O}_{K_v}^*) & v \text{ finite} \end{cases}$

The topology of the idèles

S = finite set of places

$I_K = \bigcup_S I_{K,S}$ $\quad I_{K,S} = \prod_{v \in S} K_v^* \times \prod_{v \notin S} \mathcal{O}_{K_v}^*$

(if S contains all infinite places)

$\mathbb{A}_{K,S}$ = "adelic S-integers" $\quad \mathbb{A}_{K,S} \cap K = \mathcal{O}_{K,S}$

$I_{K,S}$ = "adelic S-units" $\quad I_{K,S} \cap K^* = \mathcal{O}_{K,S}^*$

Topologize I_K as restricted direct product.

(so each $I_{K,S}$ is open): it is a locally compact topological group.

The topology of the idèles (warning!)

$\mathbb{I}_K \hookrightarrow \mathbb{A}_K$ is a continuous map, but $\mathbb{I}_K$ does ~~not~~ carry the subspace topology!

(e.g. $\mathbb{I}_{K,S}$ is open in $\mathbb{I}_K$, but is not intersection of $\mathbb{I}_K$ with an open subset of $\mathbb{A}_K$.)

Fix: view $\mathbb{I}_K$ as $GL_1(\mathbb{A}_K)$

topologize via $\mathbb{I}_K \longrightarrow \mathbb{A}_K \times \mathbb{A}_K$

$x \longmapsto (x, x^{-1})$

(compare: can topologize $K_v^\times$ via same embeddings.)

(can "cheat" using $K_v^\times \to K_v$ b/c this has open image.)

The idèle class group

$K^* \hookrightarrow I_K$ image = principal idèles

$C_K = I_K / K^*$ idèle class group

$$\begin{array}{ccc} I_K & \longrightarrow & J_K \\ (x)_v & \longmapsto & \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})} \\ \downarrow & & \\ C_K & \longrightarrow & J_K / P_K = Cl(K) \end{array}$$

J_K ← fractional ideals of K

← finite product!

The idèle class group and the class group

$$\begin{tikzcd} I_K \arrow[r, two heads] \arrow[d] & J_K \arrow[d] \\ C_K \arrow[r, two heads] & J_K/P_K = Cl(K) \end{tikzcd}$$

$$(x)_v \longrightarrow \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}$$

finite product!

What's the kernel of this map?

Generalized ideal class groups

Let $\underline{m}$ be a formal product of places

$U \subset I_K$ $\quad U = \{ (x_v)_v : v \in \underline{m} \text{ real} \Rightarrow x_v > 0$; $\quad \mathfrak{p} = v \in \underline{m}$ finite, $\mathfrak{p}^e \| \underline{m} \Rightarrow x_v \equiv 1 \bmod (\mathfrak{p}^e) \}$

$$I_K / K^* U \xrightarrow{\sim} Cl^{\underline{m}}(K) = \frac{J_K^{\underline{m}}}{P_K^{\underline{m}}}$$

ray class group

$J_K^{\underline{m}}$ ← fractional ideals coprime to $\underline{m}$

$P_K^{\underline{m}}$ ← ones with a generator in U

Compactness of the norm-1 class group

Norm map $|\ |: I_K \to \mathbb{R}^+$

$$(x_v) \mapsto \prod_v |x_v|_v$$

normalized as in product formula

$$(x_v) \mapsto C_K \to \prod_v |x_v|_v$$

Product formula:

Let $C_K^0 = \ker(C_K \to \mathbb{R}^+)$: then $C_K^0 \twoheadrightarrow Cl(K)$

prop: C_K^0 is compact. (analogue of A_K/K compact)

pf sketch: Can cover C_K^0 with a compact subset of I_K:

$$\prod_v \{x \in K_v^* : |x|_v \in [\tfrac{1}{c}, c]\}$$

for some $c > 1$. this is compact

Finiteness of the class group

C_K^0 compact $\Longrightarrow$ finiteness of the class group.

Pf C_K^0 compact. $C_K^0 \twoheadrightarrow Cl(K)$ continuous for discrete topology on $Cl(K)$

$\Longrightarrow Cl(K)$ is compact.

$\Longrightarrow$ Cor $\exists$ finite sets of places of K

s.t. $I_K = I_{K,S} \cdot K^*$ Trick $S \ni \{\mathfrak{p}\}$ generators of $Cl(K)$

The unit group (and S-unit group)

Cor compact $\Rightarrow$ for any finite set S containing all infinite places, $\exists$ an exact sequence

$$1 \to \mu_K \to \mathcal{O}_{K,S}^{*} \longrightarrow \mathbb{Z}^{\#S-1} \to 1$$

S-units

pf $\log : I_{K,S} \longrightarrow \mathbb{R}^{\#S}$ $\quad x \longmapsto \prod_{v \in S} \log|x_v|_v$

$\mathcal{O}_{K,S}^{*} \hookrightarrow H$ = trace-zero hyperplane

need to show $\mathcal{O}_{K,S}^{*}$ is discrete & cocompact in H.

The volume of the idèle class group

Choose this for S large enough so that

$$I_K = I_{K,S} \cdot K^* \Rightarrow C_K = \bar{I}_{K,S} / O_{K,S}^*$$

set $C_K^0 \longrightarrow \mathbb{H} / \mathbb{R}\text{-span}(O_{K,S}^*)$
homomorphism with compact image
which generates the target.

For $K = \mathbb{Q}$ $\quad C_{\mathbb{Q}} = \mathbb{R}^* \times \prod_p \mathbb{Z}_p^*$

For K general, we define a natural volume measure on C_K so that volume of C_K^0 = $\# Cl(K) \cdot$ unit regulator

One more point about GL_1

$$I_K = GL_1(\mathbb{A}_K)$$

$\wr\ C_K$

1-dim reps of $Gal(\bar{K}/K)$ $\longleftrightarrow$ reps of $GL_1(\mathbb{A}_K)$

change these to $n > 1$
(Langlands program)