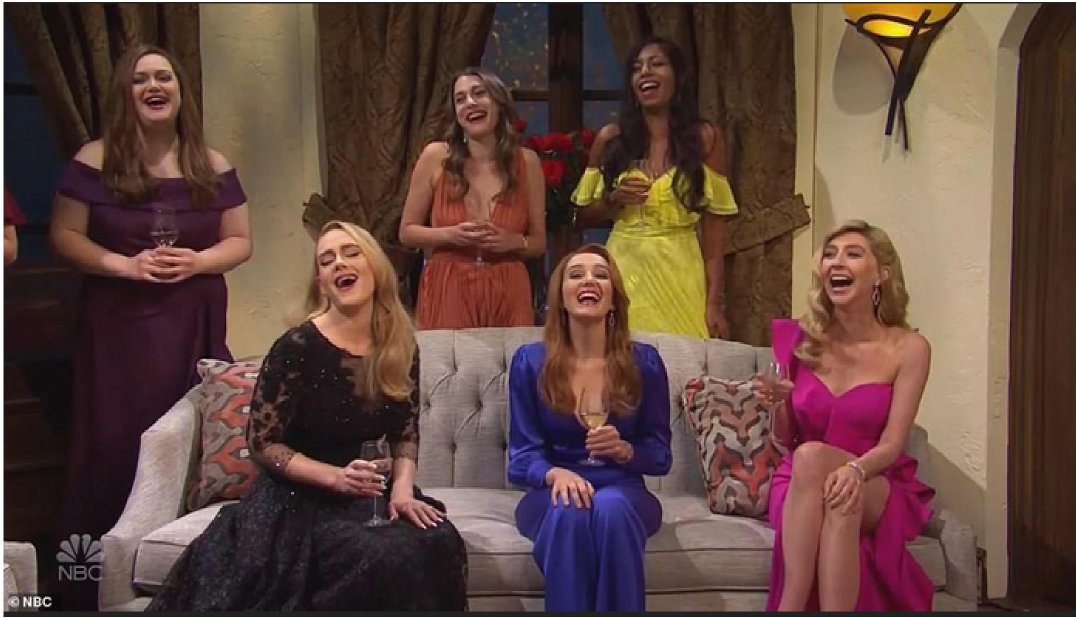
Adèles and idèles in field extensions

Adele tries a "field extension" to comedy: *Saturday Night Live*, October 24, 2020



Adèles in field extensions

L/K finite extension of # fields

note $\mathbb{A}_K = \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K$ (exercise)

$$\begin{tikzcd} & K \arrow[r, hook] \arrow[d, hook] & L \arrow[d, hook] \\ I_K \arrow[d, hook] & I_K \arrow[l, phantom] \arrow[r, hook] \arrow[d, hook] & I_L \arrow[d, hook] \\ & \mathbb{A}_K \arrow[r, hook] & \mathbb{A}_L \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}_K \arrow[d, hook] & \mathbb{A}_L \arrow[d, hook] & \prod_{v \text{ finite}} \mathcal{O}_{K_v} \arrow[r] \arrow[dll] & \prod_{w \text{ finite}} \mathcal{O}_{L_w} \arrow[dll] \\ \prod_v K_v \arrow[r] & \prod_w L_w & & \end{tikzcd}$$

$$(x_v)_v \longmapsto (x_v)_w$$

where $v = w|_K$

Automorphisms of adèle rings

$g \in \mathrm{Aut}(L/K)$

g acts on $\mathbb{A}_L$ and $I_L = \mathbb{A}_L^*$

$\mathbb{A}_L \subseteq \prod L_w$

$g \circlearrowright w$

$|a^g|_{w^g} = |a|_w$

$g : L_w \longrightarrow L_{w^g}$

If L/K not Galois, M/K Galois closure, any $g \in \mathrm{Gal}(M/K)$ carries L to $L^g \subseteq M$, $\mathbb{A}_L$ to $\mathbb{A}_{L^g} \subseteq \mathbb{A}_M$.

$\mathbb{A}_L = \mathbb{A}_K \otimes_K L$

$\circlearrowleft$

$\mathrm{Aut}(L/K)$

$\mathrm{Aut}(L/K) = \mathrm{Aut}(\mathbb{A}_L/\mathbb{A}_K)$

ignoring topology!

$\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{C}$

have trivial aut group

Trace and norm

L/K extension of # fields

Trace L/K:

$$\begin{tikzcd} L \arrow[r] \arrow[d, hook] & K \arrow[d, hook] \\ A_L \arrow[r] \arrow[d, two heads] & A_K \arrow[d, two heads] \\ A_L/L \arrow[r] & A_K/K \end{tikzcd}$$

Trace L/K :

M/K = Galois closure

$$\mathrm{Trace}_{L/K}(x) = \sum_g x^g$$

g runs over coset reps of $\mathrm{Gal}(M/L)$ in $\mathrm{Gal}(M/K)$

Norm L/K:

$$\begin{tikzcd} L^* \arrow[r] \arrow[d] & K^* \arrow[d] \\ I_L \arrow[r] \arrow[d, two heads] & I_K \arrow[d, two heads] \\ C_L \arrow[r] & C_K \end{tikzcd}$$

idele class group $\to C_L$

$$\mathrm{Norm}_{L/K}(x) = \prod_g x^g$$

Trace and norm: alternate interpretation

- $\left(\mathrm{Trace}_{L/K}(\alpha)\right)_v = \sum_{w|v} \mathrm{Trace}_{L_w/K_v}(\alpha_w)$
- $\left(\mathrm{Norm}_{L/K}(\alpha)\right)_v = \prod_{w|v} \mathrm{Norm}_{L_w/K_v}(\alpha_w)$

- Trace of α is trace of multiplication by α as an $\mathbb{A}_K$-linear endomorphism of $\mathbb{A}_L$ (finite free $\mathbb{A}_K$-module).

Similarly, norm of α is determinant of α as an $\mathbb{A}_K$-linear endomorphism of $\mathbb{A}_L$.

Invariants of the Galois action

$G = \mathrm{Gal}(L/K)$

Let L/K be a Galois extension of # fields

then $\mathbb{A}_L^G = \mathbb{A}_K$ $\quad I_L^G = I_K$.

<u>pf</u> observe that $\mathbb{A}_K = \mathbb{A}_\mathbb{Q} \otimes_\mathbb{Q} K$

<u>Of</u> $\alpha \in \mathbb{A}_L^G$ for each place v of K,

- for $w|v$, α_w is fixed by G_w = decomp group $\cong \mathrm{Gal}(L_w/K_v)$
  so it lies in K_v.
- These are all the <u>same</u> element of K_v.

The Galois action on idèles class groups

$$\begin{tikzcd}
{} \arrow[d, dashed] & {} \arrow[d, dashed] \\
K \arrow[r, hook] \arrow[d] & L \arrow[d] \\
I_K \arrow[r, hook] \arrow[d] & I_L \arrow[d] \\
C_K \arrow[r, hook] \arrow[d, dashed] & C_L \arrow[d, dashed] \\
{} & {}
\end{tikzcd}$$

$$L^G = K$$

if $G = Gal(L/K)$

$$C_K \to C_L^G$$

$$\boxed{\begin{array}{l} \mathbb{A}_K \cap L = K \\ (*) \text{ inside } \mathbb{A}_L \\ \Downarrow \\ I_K \cap L^* = K^* \text{ in } I_L \end{array}}$$

Galois descent for idèle class groups

$1 \to L^* \to I_L \to C_L \to 1$

Thm $C_K \cong C_L^G$

Pf: Look at G-cohomology of the sequence:

$$1 \to K^* \to I_K \to C_L^G \to H^1(G, L^*) \to$$

vanishes by Theorem 90

$$\Rightarrow C_K \cong C_L^G.$$

Comparison with ideal class groups

By comparison,

$$Cl(K) \longrightarrow Cl(L)^G$$

is neither injective

nor surjective in general!

Preview: the adelic reciprocity law

Thm (Adelic reciprocity)

For any Galois extension L/K of ~~A~~ fields, there is a canonical isomorphism

$$\begin{array}{ccc} C_K & \longrightarrow & \mathrm{Gal}(K^{ab}/K) \\ \downarrow & & \downarrow \\ C_K / \mathrm{Norm}_{L/K} C_L & \xrightarrow{\sim} & \mathrm{Gal}(L/K)^{ab} \end{array}$$

and $\mathrm{Norm}_{L/K} C_L$ is open in C_K.

Preview: the adelic existence theorem

For every # field K, every open subgroup H of finite index in C_K, there is a unique finite abelian extension L/K s.t. $H = \mathrm{Norm}_{L/K} C_L$.

Preview: the adelic norm limitation theorem

L/K any extension of # fields

$\star$ M/K = maximal abelian subextension

then $\mathrm{Norm}_{L/K}\, \mathbb{C}_L = \mathrm{Norm}_{M/K}\, \mathbb{C}_M$.

inside $\mathbb{C}_K$

Local-Global compatibility $G = Gal(L/K)$

L/K be an abelian extension of # fields

Let v = place of K, w = place of L above v

$G_v \cong Gal(L_w/K_v)$

Define: $r_{K,v} : K_v^* \longrightarrow G_v \hookrightarrow G$

- if v is finite, use local reciprocity map.
- if v is real $K_v^* = \mathbb{R}^* \xrightarrow{\text{sign}} \{\pm 1\} = G_v \hookrightarrow G$
- if v is complex $K_v^* \longrightarrow G_v = \{1\} \longrightarrow G$

Local-global compatibility

$$\tilde{r}_K : I_K \to G \qquad (\alpha_v) \mapsto \prod_v r_{K_v}(\alpha_v).$$

Note: for almost all v, L/K is unramified above v and $\alpha_v \in \mathcal{O}_{K_v}^\times$, so $r_{K_v}(\alpha_v) = 1$.

$\boxed{\text{Prop}}$ The map $\tilde{r}_K$ factors through C_K iff. $\tilde{r}_K(K^\times) = \{e\}$.