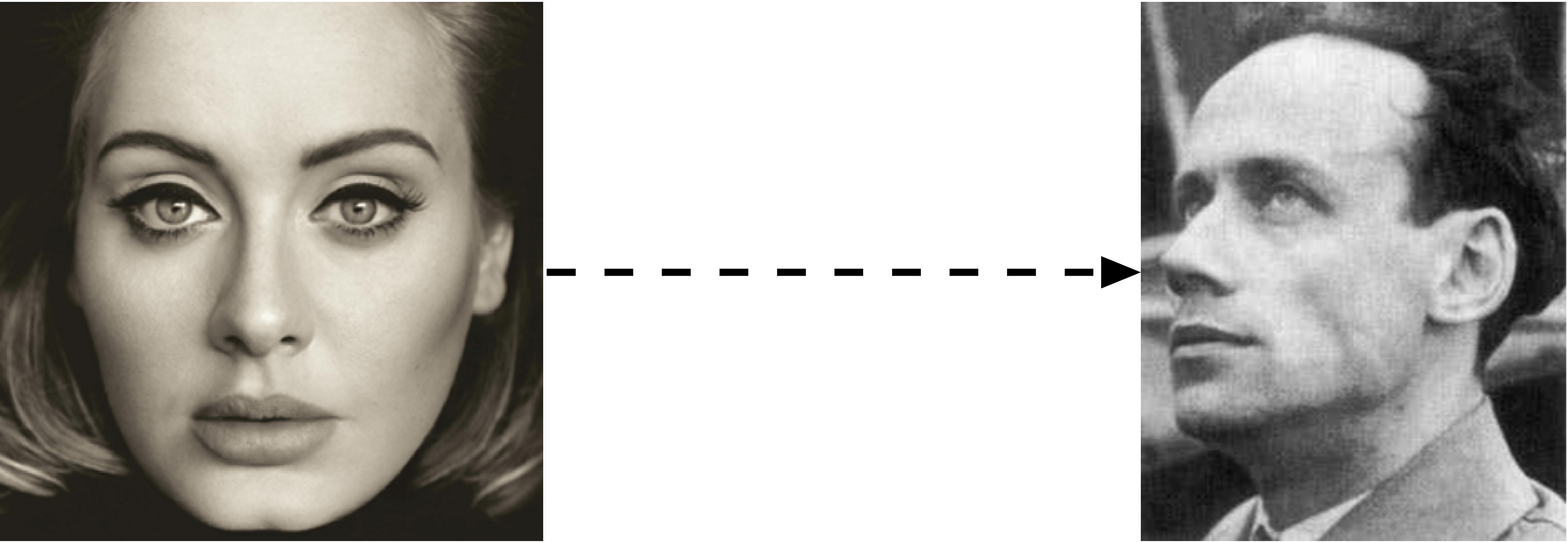
Artin reciprocity via adelic reciprocity

HW6 is posted.



Statement of the Artin reciprocity law

Let L/K be an abelian extension of # fields.

<u>A Artin reciprocity:</u> For some product $\mathfrak{m}$ of places of K,

the Artin map $J_K^{\mathfrak{m}} \longrightarrow \mathrm{Gal}(L/K)$

factors through

$$\begin{tikzcd} J_K^{\mathfrak{m}} \arrow[r] \arrow[d] & \mathrm{Gal}(L/K) \\ J_K^{\mathfrak{m}}/P_K^{\mathfrak{m}} = Cl^{\mathfrak{m}}(K) \arrow[ur, dotted] & \end{tikzcd}$$

e.g. $K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_n)$

i.e. the map $\underset{\text{prime of } K}{\mathfrak{p}} \longrightarrow \mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(L/K)$

is determined by a <u>congruence condition</u> on $\mathfrak{p}$.

Statement of the adelic reciprocity law

L/K abelian

$C_K = I_K / K^*$ idele class group $\quad C_L = I_L / L^*$

Adelic reciprocity: there exists a canonical isomorphism

$$\mathrm{Gal}(L/K) \cong C_K / \mathrm{Norm}_{L/K} C_L$$

if L' contains L, then

$$\begin{tikzcd}
\mathrm{Gal}(L'/K) \arrow[r, "\sim"] \arrow[d] & C_K / \mathrm{Norm}_{L'/K} C_{L'} \arrow[d] \\
\mathrm{Gal}(L/K) \arrow[r, "\sim"] & C_K / \mathrm{Norm}_{L/K} C_L
\end{tikzcd}$$

Local reciprocity maps

L/K = abel. extension of # fields

v = place of K, w = place of L above v

$G_v \cong \mathrm{Gal}(L_w/K_v) \subseteq G = \mathrm{Gal}(L/K)$

$r_{K,v} : K_v^* \longrightarrow G_v \hookrightarrow G$

local reciprocity map if v is finite

(or $\mathbb{R}^* \longrightarrow \{\pm 1\}$ sign
$\mathbb{R}^* \longrightarrow \{1\}$
$\mathbb{C}^* \longrightarrow \{1\}$)

Local-global compatibility

$$\tilde r_K : I_K \twoheadrightarrow G \qquad \tilde r_K((\alpha_v)_v) = \prod_v r_{K,v}(\alpha_v)$$

compatibility: this factors through

$$\begin{tikzcd}
I_K \arrow[r, two heads, "\tilde r_K"] \arrow[d, bend right] & G \\
I_K/K^* = C_K \arrow[ur, dashed, "r_K"'] &
\end{tikzcd}$$

$$r_K : C_K \to G$$

and this induces the isomorphism

$$C_K/\mathrm{Norm}_{L/K}\, C_L \cong G \quad \text{from before.}$$

Adelic reciprocity + local global = Artin reciprocity

L/K be an abelian extension of # fields.

Let $\mathfrak{p}$ be a prime of K $\quad \pi \in K$ uniformizer at $\mathfrak{p}$

if $\mathfrak{p}$ unramified in L

$\alpha = (1, \ldots, \pi, 1, \ldots)$ $\quad \alpha_v = \begin{cases} \pi & \text{if } v = \mathfrak{p} \\ 1 & \text{else} \end{cases}$

Via local reciprocity this maps

to $\mathrm{Frob}_{\mathfrak{p}} \in G_v \subseteq G$.

Via $\mathbb{I}_K \to C_K \twoheadrightarrow C_K / \mathrm{Norm}_{L/K} C_L \to G$

$\cong$

$Cl^m(K)$

$\uparrow$ open subgroup of finite index

Refinements of Artin reciprocity

This derivation also implies:

- the conductor $\mathfrak{m}$ of L/K is divisible only by ramified primes.
- kernel of classical norm map is generated by principal ideals $\equiv 1 \bmod \mathfrak{p}$ and Norms of ideals of L coprime to $\mathfrak{D}_{L/K}$.

Example: cyclotomic extensions

e.g. $K = \mathbb{Q}$

$L = \mathbb{Q}(\zeta_n)$ $\quad \mathfrak{p} = (p)$ $\quad$ say $p \nmid n$

$$
\begin{tikzcd}
 & K_v^* = \mathbb{Q}_p^* \ni \pi = p \arrow[rr] \arrow[d, "{(1,\ldots,\pi,1,\ldots)}"] & & \mathrm{Gal}(L_w/K_v) \cong \langle p \rangle \subseteq (\mathbb{Z}/n\mathbb{Z})^* \arrow[d, hook] \\
K^* \arrow[ur] \arrow[r, "\mathrm{diag}"'] & C_K / N_{L/K} C_L \arrow[rr] & & \mathrm{Gal}(L/K) \cong (\mathbb{Z}/n\mathbb{Z})^*
\end{tikzcd}
$$

Reminder: the setup of abstract CFT

k = field

$G = \mathrm{Gal}(\bar{k}/k) \twoheadrightarrow \hat{\mathbb{Z}}$ "unramified quotient"

$e_{L/K}$ $f_{L/K}$

$A = G\text{-module} = \bigcup_{K/k \text{ finite}} A^{G_K}$ $G_K = \mathrm{Gal}(\bar{k}/K)$

- For L/K cyclic extension of finite exts of k,

$$H_T^i(\mathrm{Gal}(L/K), A_L) = \begin{cases} [L:K] & i = 0 \\ 0 & i = 1 \end{cases}$$

$v = A_k \longrightarrow \hat{\mathbb{Z}}$ "Henselian valuation"

$v(\mathrm{Norm}_{K/k} A_K) = f_{K/k}\, v(A_K)$

What needs to be done: the First Inequality

Want to apply abstract CFT for

$k = \mathbb{Q}$

$A = \bigcup_{K/k \text{ finite}} C_K$

$d, v = ?$

w/ L/K cyclic extension of # fields.

compute $h(C_L) = [L:K]$ (a la Dirichlet unit theorem).

$$\Downarrow$$

$$\# H^0_T(\mathrm{Gal}(L/K), C_L) \geq [L:K]$$

First Inequality

What needs to be done: the Second Inequality

Second Inequality: $\# H^0_T(\mathrm{Gal}(L/K), C_L) \leq [L:K]$

NOT CLEAR how to show directly that either $H^0(\mathrm{Gal}(L/K), C_L)$ or $H^{-1}_T(\mathrm{Gal}(L/K), C_L)$ is trivial.

Two approaches

- analytic argument (Dirichlet density)

- algebraic approach, using Kummer theory (similar to existence theorem)

What needs to be done: the abstract reciprocity law

Apply abstract C-T with $k = \mathbb{Q}$

$A = \varinjlim_K C_L$

$d : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}^{smcy}/\mathbb{Q})$ small cyclotomic

$\cong \hat{\mathbb{Z}}^* \twoheadrightarrow \hat{\mathbb{Z}}$

$v : C_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q})$

$\| \qquad \prod_p \mathbb{Z}_p^* \longrightarrow \prod_p \mathbb{Z}_p$

$\downarrow \qquad \mathbb{Z}_p^*/(\text{torsion}) \xrightarrow[\sim]{\log} \mathbb{Z}_p$ ← choose!!!

some artificial choice of isom $\mathrm{Gal}(\mathbb{Q}^{smcy}/\mathbb{Q}) \cong \hat{\mathbb{Z}}$

$\cong \hat{\mathbb{Z}}^*/\text{torsion}$

to define d and v $\Rightarrow$ resulting isom $r_{L/K} : \mathrm{Gal}(L/K)^{ab} \cong C_K/N_{L/K}C_L$ is well-defined!!

What needs to be done: local-global compatibility

Work with totally ramified cyclotomic extensions to get a local-global compatibility

(key: every abelian ext of $\mathbb{Q}_p$ by local Kronecker–Weber, is completion of an abelian extension of $\mathbb{Q}$!)

Another approach: Galois cohomology

Can also prove global CFT

using "plain A" —

direct computation in Galois cohomology

(especially $H^2(\mathrm{Gal}(L/K), L^*)$

and $H^2(\mathrm{Gal}(\bar{K}/K), \bar{K}^*)$

= Brauer group of K.

Parting thoughts (as time permits)

see last lecture.