Cohomology of the idèles I: the “First Inequality”

Hereditary inequality dates back to the Stone Age

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Hereditary inequality began over 7,000 years ago in the early Neolithic era, with new evidence showing that farmers buried with tools had access to better land than those buried without.

The research, carried out by archaeologists from the Universities of Bristol, Cardiff, Oxford and Durham, is published in PNAS today [28 May].
By studying more than 300 human skeletons from sites across central Europe, Professor Alex Bentley and an international team of colleagues funded by the Arts and Humanities Research Council uncovered evidence of differential land access among the first Neolithic farmers – the earliest such evidence yet found.

Strontium isotope analysis of the skeletons, which provides indications of place of origin, indicated that men buried with distinctive Neolithic stone adzes (tools used for smoothing or carving wood) had less variable isotope signatures than men buried without adzes. This suggests those buried with adzes had access to closer – and probably better – land than those buried without.

Professor Bentley, Professor of Archaeology and Anthropology at the University of Bristol, said: “The men buried with adzes appear to have lived on food grown in areas of loess, the fertile and productive soil favoured by early farmers. This indicates they had consistent access to preferred farming areas.”

https://www.bristol.ac.uk/news/2012/8537.html

Reminder: the class field axiom

For abstract CFT, we need:

For L/K a cyclic extension of # fields,

$$\# \hat{H}_T^i(\mathrm{Gal}(L/K), \underbrace{C_L}_{\text{idele class group}}) = \begin{cases} [L:K] & i \text{ even } (i=0, i=2) \\ 1 & i \text{ odd } (i=1, i=-1) \end{cases}$$

$$C_L = \mathbb{I}_L / L^\times$$

$$\mathbb{I}_L = \mathbb{A}_L^\times$$

The Herbrand quotient and the First Inequality

Today: compute $\left[h(C_L) = \dfrac{\# H^0_T(\mathrm{Gal}(L/K), C_L)}{\# H^1_T(\mathrm{Gal}(L/K), C_L)} = [L:K] \right.$

$\Longrightarrow \# H^0_T(\mathrm{Gal}(L/K), C_L) \geq [L:K]$.

"First Inequality"

A direct sum decomposition for cohomology

L/K finite Galois extension $\quad G = \mathrm{Gal}(L/K)$

S = finite set of places of K containing all infinite places

$I_{L,S} = I_{L,T}$

$$\prod_{w \in T} L_w^* \times \prod_{w \notin T} \mathcal{O}_{L_w}^*$$

$T = \{w \mid v : v \in S\}$ places of L

$I_L = \bigcup_S I_{L,S}$

$\circlearrowleft G \qquad \circlearrowleft G$

For S sufficiently large

$$\boxed{I_L = I_{L,S} L^*}$$

A direct sum decomposition for cohomology: proof

Prop (1) a $\quad H^i(G, I_L) = \bigoplus_v H^i(G_w, L_w^*)$

(w some place of L above v)

($i \in \mathbb{Z}$) $\quad H_T^i(G, I_L) = \bigoplus_v H_T^i(G_w, L_w^*)$

Pf $\quad H^i(G, I_L) = \varinjlim_S H^i(G, I_{L,S})$

assume S contains all ramified places

$$= \varinjlim_S \bigoplus_{v \in S} \underbrace{H^i\Big(G, \prod_{w|v} L_w^*\Big)}_{\substack{\text{Shapiro's lemma} \\ = H^i(G_w, L_w^*)}} \times \prod_{v \notin S} \underbrace{H^i\Big(G, \prod_{w|v} \mathcal{O}_{L_w}^*\Big)}_{\text{trivial.}}$$

H^1 and H^2

<u>(w</u> $H^1(G, I_L) = 0$

$$H^2(G, I_L) \cong \bigoplus_v \frac{1}{[L_w : K_v]}\mathbb{Z}/\mathbb{Z}$$

local CFT for v finite

but also OK if v is ~~comp~~ complex

if v real and $G_w = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$

$$H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^*) \cong \hat{H}^0_T(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^*)$$

$$\cong \mathbb{R}^* / \mathrm{Norm}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^* = \mathbb{R}^* / \mathbb{R}^{*+} \cong \{\pm 1\}$$

The norm subgroup is open

S contains all infinite places, all ramified places, and is big enough that $I_L = I_{L,S} \cdot L^*$

$$\mathrm{Norm}_{L/K} I_{L,S} = \prod_{v \in S} U_v \times \prod_{v \notin S} O_{K_v}^*$$

for some U_v open of finite index in K_v^*

$\Rightarrow$ <u>$\mathrm{Norm}_{L/K} C_L$ is open</u> of finite index in C_K.

$$\Rightarrow \quad C_K \cong I_K / (K^* \cdot \mathrm{Norm}_{L/K} I_L)$$

Cohomology of the units: first steps

$\overline{I}_L = I_{L,S} \cdot L^*$

Now assume $G = \mathrm{Gal}(L/K)$ cyclic.

$$1 \to \mathcal{O}_{L,S}^* \to I_{L,S} \to C_L \to 1 \qquad \boxed{G_v = G_w}$$

$$\Rightarrow h(I_{L,S}) = h(\mathcal{O}_{L,S}^*)\, h(C_L)$$

$$= \prod_{v \in S} h(G_v, L_w^*) = \prod_{v \in S} \# H_T^0(G_v, L_w^*)$$

$$= \prod_{v \in S} [L_w : K_v]$$

So it suffices to show: $\boxed{h(\mathcal{O}_{L,S}^*) = \frac{1}{[L:K]} \prod_{v \in S} [L_w : K_v].}$

A computation with S-units

$\curvearrowright G$ $\qquad$ $G \circlearrowleft$ $\qquad$ $V = \prod_{w \in T} \mathbb{R}$ $\qquad$ $T = \{w \mid v : v \in S\}$

$$\mathcal{O}_{L,S}^* \longrightarrow V \qquad \alpha \longmapsto (\log|\alpha|_w)_w$$

image is a lattice M in H = trace-zero hyperplane in V.
$\circlearrowleft G$

$M_1 :=$
$M + \langle (1, \dots, 1) \rangle$ is a lattice in V.
G-stable

$$h(M_1) = h(M)\underbrace{h(\mathbb{Z})} = (L:K).$$

We need: $h(M_1) = \prod_{v \in S} [L_w : K_v]$.

Comparison of lattices

$M_2 = \prod_{v \notin T} \mathbb{Z} \circlearrowleft G$ is lattice in V

with $h(G, M_2) = \prod_{v \in S} h(G, \mathrm{Ind}_{G_v}^{G} \mathbb{Z})$

$= \prod_{v \in S} h(G_v, \mathbb{Z})$

$= \prod_{v \in S} \# G_v = \prod_{v \in S} (L_w : K_v)$

$\prod$

Lemma: if M_1, M_2 are two G-stable lattices in same $\mathbb{R}$-vectorspace, then $h(M_1) = h(M_2)$

Comparing Herbrand quotients of lattices

Pf If M_1, M_2 are commensurable (i.e.

$M_1 \cap M_2$ has finite index in both), then

$h(M_1) = h(M_1 \cap M_2) = h(M_2)$.

Point In general, $M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_2 \otimes_{\mathbb{Z}} \mathbb{Q}$

are $\mathbb{Q}$-linear reps of which become isomorphic

over $\mathbb{R}$ $\Rightarrow$ already isomorphic over $\mathbb{Q}$!

Lemma: G = finite group, F/E extension of (infinite) fields

V_1, V_2 are finite-dim E-vector spaces with E-linear (right) G-action,

s.t. $V_1 \otimes_E F \cong V_2 \otimes_E F$ as G-reps.

Then $V_1 \cong V_2$.

Proof of the lemma $\qquad T \quad T^g(x) = T(x^{g^{-1}})^g$ (adjoint action)

$$G \circlearrowright W = \mathrm{Hom}_E(V_1, V_2) = V_1^\vee \otimes_E V_2$$

$$G \circlearrowright W_F = W \otimes_E F \cong \mathrm{Hom}_F(V_1 \otimes_E F, V_2 \otimes_E F)$$

$$W^G \otimes_E F \cong W_F^G$$ ← assumed this contains an isomorphism.

That is: if I fix an identification of underlying vector spaces $V_1 \cong V_2$ (not G-isomorphism!), determinant function is not identically zero on W_F^G, hence not identically zero on W^G either (b/c E infinite).

Corollary: splitting of primes

For L/K any nontrivial extension of $\#$ fields, $\exists$ infinitely many primes of K that do not split completely in L.

Pf If L/K is cyclic of prime order and all but finitely many primes split completely, put the rest into S.
Set $C_K = \mathrm{Norm}_{L/K} C_L$, contradicting First Inequality:
$\# C_K / \mathrm{Norm}_{L/K} C_L \geq [L:K]$.

General case: Let M/K be Galois closure of L/K.

$\mathfrak{p} \subset K$ splits completely in $L \iff$ splits completely in M

So assume L/K is ~~Galois~~ nontrivial.

So $\mathrm{Gal}(L/K) \supseteq$ nontrivial cyclic group of prime order

(corresponding to

$L \overset{\text{cyclic, prime order}}{—} K' — K$

Apply previous case to L/K!