Hereditary inequality dates back to the Stone Age

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Hereditary inequality began over 7,000 years ago in the early Neolithic era, with new evidence showing that farmers buried with tools had access to better land than those buried without.

The research, carried out by archaeologists from the Universities of Bristol, Cardiff, Oxford and Durham, is published in PNAS today [28 May]. By studying more than 300 human skeletons from sites across central Europe, Professor Alex Bentley and an international team of colleagues funded by the Arts and Humanities Research Council uncovered evidence of differential land access among the first Neolithic farmers – the earliest such evidence yet found.

Strontium isotope analysis of the skeletons, which provides indications of place of origin, indicated that men buried with distinctive Neolithic stone adzes (tools used for smoothing or carving wood) had less variable isotope signatures than men buried without adzes. This suggests those buried with adzes had access to closer – and probably better – land than those buried without.

Professor Bentley, Professor of Archaeology and Anthropology at the University of Bristol, said: “The men buried with adzes appear to have lived on food grown in areas of loess, the fertile and productive soil favoured by early farmers. This indicates they had consistent access to preferred farming areas.”

https://www.bristol.ac.uk/news/2012/8537.html
Reminder: the class field axiom

For abstract KGT, we need:

For $L/K$ a cycl. extension of fields,

$$\frac{1}{i} \left( \text{val} (L/K), C_L \right) = \bigcup \left( L/K \right)^{\circ \circ}_{i \in \text{even}} \times \text{class group} \bigg\{ 1 \bigg| i \text{ odd} \bigg\}$$

$$C_L = \text{Gal}(L \cap K)$$

$A^1_L$
The Herbrand quotient and the First Inequality

\[ \text{Ludmy: compute } \sqrt{h(C)} = \frac{\# M^0(\text{cal}(C/K), C)}{\# M^0(\text{cal}(C/K), C)} = (L:K) \]

\[ \Rightarrow \# H^0_{-1}(\text{cal}(C/K), C/L) \approx (L:K) \]

"First Inequality"
A direct sum decomposition for cohomology

\[ \mathbb{L}/K \text{ finite Galois extension} \]
\[ S = \text{ a set of primes of } K \]
\[ L, S = \bigcap_{\mathfrak{p} \nmid \mathfrak{m}_s} \mathfrak{p} \]

\[ T = \bigoplus \mathbb{Q}_c \]

\[ I_L = \bigcap_{\mathfrak{p} \nmid \mathfrak{m}_S} L^* \]
A direct sum decomposition for cohomology: proof

\[
\text{Proof}\quad \check{H}^i(C_\mathcal{L}) = \bigoplus \hat{H}^i(G_w, \mathcal{L}^\vee_w)
\]

\[
(C_\mathcal{L}) \Rightarrow (G, \mathcal{L}) = \bigoplus \hat{H}^i(G_w, \mathcal{L}^\vee_w)
\]

\[
\Rightarrow H^i(G, \mathcal{L}) = \bigoplus \hat{H}^i(G_w, \mathcal{L}^\vee_w)
\]

Assume $S$ contains all ramified places.
\[ H^1 (U, \mathbb{C}) = 0 \]

\[ H^2 (U, \mathbb{C}) \cong \bigoplus \frac{1}{\sqrt{L(w_i)R}} / \mathbb{Z} \]

Local for finite set also OK if \( V \) is compact.

If \( V \) real and \( \mathcal{O}_W = \mathcal{O}_{\text{alb}} (\mathcal{O}_{\text{alb}}) \)

\[ H^2 (\mathcal{O}_{\text{alb}} (\mathcal{O}_{\text{alb}}), \mathbb{C}^*) \cong H^0_+ (\mathcal{O}_{\text{alb}} (\mathcal{O}_{\text{alb}}), \mathbb{C}^*) \cong \mathbb{R}^* / \mathbb{Z}_{\mathbb{R}} \]

\[ \mathcal{C}^* = \mathbb{R}^* / \mathbb{Z}^* \]
The norm subgroup is open

Since this all in finite place, all finite places, and it is enough that \( \Gamma_{L,v} \subset K^* \) for some \( v \in \mathcal{V} \).

\[
\text{for some } v \text{ open of finite index in } K^* \text{ in } \mathcal{V},
\]

\[
\text{if norm } \Lambda_{L/K} C_L \text{ is open of finite index in } C_L.
\]

\[
\Rightarrow C_K = \frac{I_K}{(K^*, N|_{\text{norm unk } C_L})}
\]
Cohomology of the units: first steps

\[ H_2 = I_L, S \cdot L^* \]

\[ 1 \rightarrow \mathcal{O}_L^* \rightarrow I_{L, S} \rightarrow C_L \rightarrow 1 \]

\[ h(I_{L, S}) = h(\mathcal{O}_L^*) h(C_L) \]

\[ = \prod_{v \in S} h(C_v, L_v^*) = \prod_{v \in S} H^0_u(C_v, L_v^*) \]

\[ = \prod_{v \in S} \left[ L_v : K_v \right] \]

So it will suffice to show:

\[ h(\mathcal{O}_L^*) = \frac{1}{[L : K]} \prod_{v \in S} \left[ L_v : K_v \right] \]
A computation with S-units

\[ V = \bigcup_{T \in \mathbb{N}_0} \mathbb{R}^T \quad \text{wrt} \quad \mathfrak{v} \in S \]

\[ \mathfrak{o}^*_{L, S} \rightarrow V \xrightarrow{\alpha} (\log_{12} \mathfrak{v}) \quad \text{w} \]

\[ \text{Image is a lattice } M_m \quad M = \text{trace zero} \quad (O_{L} \text{ hyperplane in } V) \]

\[ M = (1, \ldots, 1) \quad \text{is a lattice in } V \]

\[ h(M) = h(M) \overline{h(2)} = (L : K) \]

\[ \text{we need: } h(M) = \prod_{\mathfrak{p} \in S} \mathfrak{p}^{h(M_{\mathfrak{p}})} \]
Comparison of lattices

\[ M_2 = \biggoplus_{\text{w.t.f.}} \mathbb{Z} \] \[ \text{is lattice in } V \]

with \[ h(G, M_2) = \prod_{s \in S} h(G, \text{Ind}_{G^s}^G \mathbb{Z}) \]

\[ = \prod_{s \in S} h(G^s, \mathbb{Z}) \]

\[ = \prod_{s \in S} \# C_s = \prod_{s \in S} (C_m \cdot L_s) \]

if \( M_1, M_2 \) are two \( G \)-stable lattices in \( V \).

If \( \langle \mathbf{z} \rangle \) is \( 12 \)-vector, \( \text{then } h(M_1) = h(M_2) \)
Comparing Herbrand quotients of lattices

If $M_1, M_2$ are commensurable (i.e., $M_1 \cap M_2$ has finite index in both $M_1, M_2$), then $h(M_1) = h(M_1 \cap M_2) = h(M_2)$.

Jointly in general, $M_1 \oplus M_2$ and $M_2 \oplus M_1$ are Z-graded vector spaces of which the sum is

\[ \bigoplus_{n \in \mathbb{Z}} R^n \]
Lemma 3.1: Let $\mathbb{G} \simeq \text{Aut}_k \Gamma$ be a group of field automorphisms of $\Gamma$. Suppose $V_1, V_2$ are finite-dimensional $\mathbb{G}$-vector spaces with $\mathbb{G}$-linear (right) $\mathbb{G}$-action.

If $V_1 \otimes E F \cong V_2 \otimes E F$ as $\mathbb{G}$-modules,

then $V_1 \cong V_2$. 

Therefore, $V_1 = V_2$. 
The lemma \( T \circ T^{-1}(x) = x \) for \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) states that the determinant function is not identically zero on \( \mathbb{R}^2 \), hence not identically zero on \( \mathbb{R}^2 \).
Corollary: splitting of primes

For \( L/K \) any non-trivial extension of fields, \( L \) unramified over \( K \) if there are no primes of \( K \) that do not split completely in \( L \).

If \( L/K \) is cyclic of prime order, and all but finitely many prime splits completely, but the rest into \( L \), set \( C_K = \text{Norm}_{L/K}(L, \text{contradicts first inequality}) \)
General case: let \( M/K \) be the closure of \( L/K \).

If \( K \) splits completely in \( L \), then \( L/K \) splits completely in \( M \).

So assume \( L/K \) is not cyclic normal.

Set \( G = G(L/K) \) to be normal cyclic group of prime order corresponding to \( L/K \).

Any remaing case to \( L/K \).