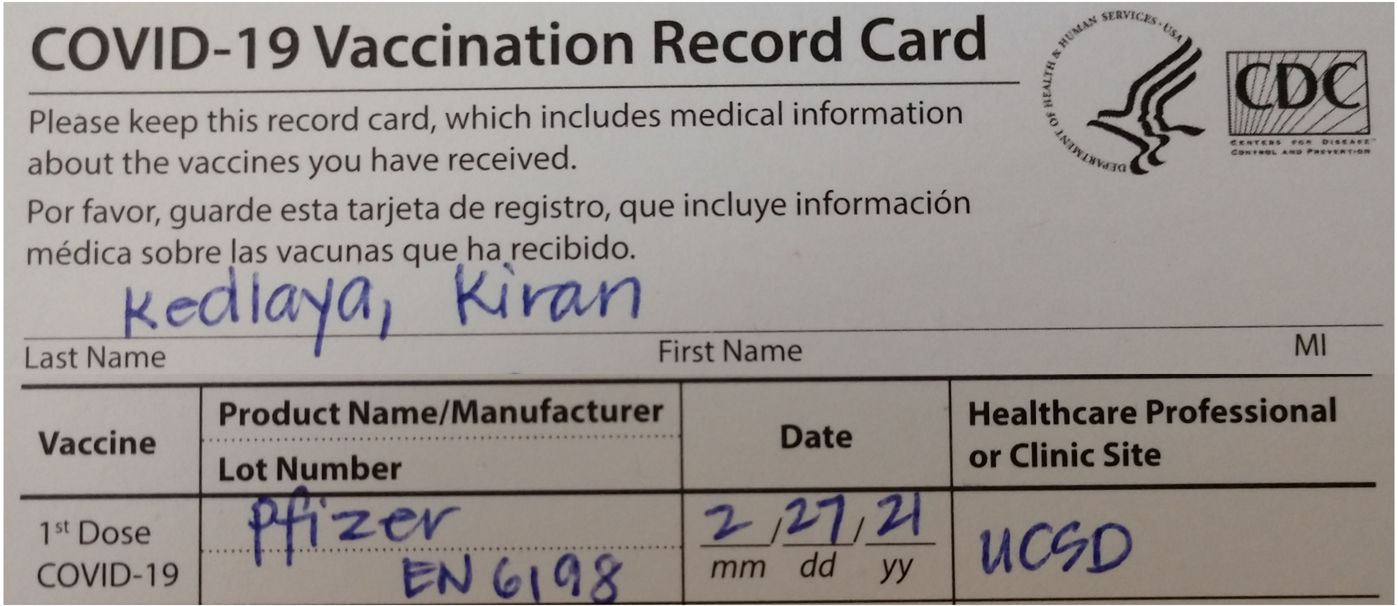
Cohomology of the idèles II: the “Second Inequality”

HW 17 has been posted.

I have been continuing to reorganize the notes. The web site has been updated so that the references for each lecture remain accurate.



Reminder: the First Inequality

L/K cyclic extension of # fields

class field axiom: $\# H^0_T(\mathrm{Gal}(L/K), C_L) = [L:K]$

$\# H^1_T(\mathrm{Gal}(L/K), C_L) = 1$

First Inequality: $h(C_L) = [L:K]$

$\Rightarrow \# \underbrace{H^0_T(\mathrm{Gal}(L/K), C_L)}_{= C_K / \mathrm{Norm}_{L/K} C_L} \geq [L:K]$

Back to the language of ideals

(ie H^1_T = trivial)

We need $\# H^0_T(\mathrm{Gal}(L/K), C_L) \le [L:K]$

For $\underline{m}$ a formal product of places of K

$J_K^{\underline{m}}$ = group of fractional ideals of K coprime to $\underline{m}$

(and $J_L^{\underline{m}}$)

$P_K^{\underline{m}}$ = subgroup of principal fractional ideals of form (α)

(and $P_L^{\underline{m}}$) where: $\alpha \equiv 1 \mod \underline{m}$ finite

$\alpha > 0$ in each real place in $\underline{m}$

$Cl^{\underline{m}}(K) = J_K^{\underline{m}} / P_K^{\underline{m}}$ $\qquad$ $Cl^{\underline{m}}(L) = J_L^{\underline{m}} / P_L^{\underline{m}}$

The Second Inequality in the language of ideals

$$I_K^{\mathfrak m} \twoheadrightarrow J_K^{\mathfrak m} \qquad \rightsquigarrow \qquad I_K^{\mathfrak m}/(K^* \cap I_K^{\mathfrak m}) \twoheadrightarrow Cl^{\mathfrak m}(K)$$

ideles → ideals

$$I_K^{\mathfrak m}/(K^* \cap I_K^{\mathfrak m}) \cong C_K \twoheadrightarrow Cl^{\mathfrak m}(K)$$

<u>Lemma</u> $C_K/\mathrm{Norm}_{L/K} C_L \xrightarrow{\sim} J_K^{\mathfrak m}/P_K^{\mathfrak m}\,\mathrm{Norm}_{L/K} J_L^{\mathfrak m}$

is an isomorphism for some $\underline{\mathfrak m}$ (really, any sufficiently divisible $\underline{\mathfrak m}$)

Pf S = finite places that ramify, in $\underline{\mathfrak m}$

For $v \in S$, $\mathrm{Norm}_{L_w/K_v} L_w^* = U_v \subset K_v^*$ open of finite index

w/ v

choose $\underline{\mathfrak m}$ so that condition $\alpha \equiv 1 \bmod \underline{\mathfrak m}$ implies $\alpha \in U_v$

$\Longrightarrow$ Second Inequality $\Longleftrightarrow$ $[J_K^{\mathfrak m} : P_K^{\mathfrak m}\,\mathrm{Norm}_{L/K} J_L^{\mathfrak m}] \le [L:K]$

A special case of Chebotarëv density

Lemma Let L/K be a Galois extension of # fields. The set S of prime ideals of K which split completely in L has Dirichlet density $\frac{1}{[L:K]}$.

$$\left(\text{i.e. } \lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in S} \mathrm{Norm}(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}} \mathrm{Norm}(\mathfrak{p})^{-s}}\right).$$

Pf In L, the set of primes of absolute degree 1 has density 1. Each prime of K which splits completely accounts for $[L:K]$ such primes of L.

A corollary For H a subgroup of J_K^m containing P_K^m of finite index

the set of primes in H has Dirichlet density either $\dfrac{1}{[J_K^m : H]}$ or 0

as $s \to 1^+$

Pf $\displaystyle\sum_{\chi : J_K^m/H \to \mathbb{C}^*} \log L(s,\chi) \approx \sum_{\chi} \sum_{p} \chi(p)\, \mathrm{Norm}(p)^{-s} + \cdots$

$\displaystyle\sim [J_m^K : H] \sum_{p \in H} \mathrm{Norm}(p)^{-s}$.

$\log L(s,\chi) \sim \overset{\text{order at } s=1}{m(\chi)} \cdot \log(s-1)$

so density in question $= \dfrac{1 - \sum_{\chi \neq 1} m(\chi)}{[J_K^m : H]}$

The Second Inequality for ideals

$$H = P_K^m \, \mathrm{Norm}_{L/K} J_L^m \subseteq J_K^m$$

Note: H contains every prime that splits completely.

So H has Dirichlet density <u>at least</u> $1/[L:K]$

But by lemma, density is ~~either 0 or~~ $\dfrac{1}{[J_K^m : H]}$.

$$\Longrightarrow [J_K^m : H] \le [L:K].$$

$\Big($for $m \gg 0$, $= [C_K : \mathrm{Norm}_{L/K} C_L]\Big)$

H^1 and H^2

L/K any Galois extension

Cor $\# \hat{H}^0_T(\mathrm{Gal}(L/K), C_L) \leq [L:K]$.

Cor $H^1(\mathrm{Gal}(L/K), C_L) = \{1\}$

$\# H^2(\mathrm{Gal}(L/K), C_L) \leq [L:K]$.

H^1 and H^2

pf For L/K cyclic, combine First & Second Inequality

For L/K solvable, use inflation-restriction to induct on $[L:K]$

$L/K'/K$

$$0 \to H^i(\mathrm{Gal}(K'/K), C_{K'}) \xrightarrow{\text{Inf}} H^i(\mathrm{Gal}(L/K), C_L) \xrightarrow{\text{Res}} H^i(\mathrm{Gal}(L/K'), C_L)$$

L/K general: compare $\mathrm{Gal}(L/K)$ with Sylow p-subgroup $\mathrm{Gal}(L/K')$

then argue that $H^i(\mathrm{Gal}(L/K), C_L)$, $H^i(\mathrm{Gal}(L/K'), C_L)$ have same p-primary components

Aside: the Hasse norm theorem

Thm L/K cyclic extension of # fields.

Then $x \in K^*$ is in $\mathrm{Norm}_{L/K} L^*$ iff for each place v of K, for w a place above v, $x \in \mathrm{Norm}_{L_w/K_v} L_w^*$.

Pf By First+Second Inequality $\hat{H}^{-1}_T(\mathrm{Gal}(\overline{L}/K), C_L) = \{1\}$.

$\Rightarrow H^0_T(G, L^*) \to H^0_T(G, I_L)$ is injective.

Rmk Can be used to prove Hasse-Minkowski theorem on quadratic forms (cf. exercise)

Rmk This fails if L/K abelian but not cyclic (exercises)

The ~~Grunwald~~ Grunwald-Wang theorem

Grunwald: For K a # field, n a positive integer,

$x \in K^*$ is is $(K^*)^n$ iff $\forall$ places v of K,

$x \in (K_v^*)^n$.

(see exercises)

This is false (Wang), but can be corrected

(e.g. true if n odd)

The Albert-Brauer-Hasse-Noether theorem

<u>Thm</u> For any # field K,

$$\underbrace{H^2(\mathrm{Gal}(\bar K/K), \bar K^*)}_{\text{Brauer group}} \hookrightarrow \bigoplus_v \underbrace{H^2(\mathrm{Gal}(\bar K_v/K_v), \bar K_v^*)}_{\text{Brauer group}},$$

<u>Pf:</u> L/K finite,

$$1 = H^1(\mathrm{Gal}(L/K), C_L) \to H^2(\mathrm{Gal}(L/K), L^*) \to H^2(\mathrm{Gal}(L/K), I_L)$$

$$\bigoplus H^2(\mathrm{Gal}(L_w/K_v), L_w^*)$$