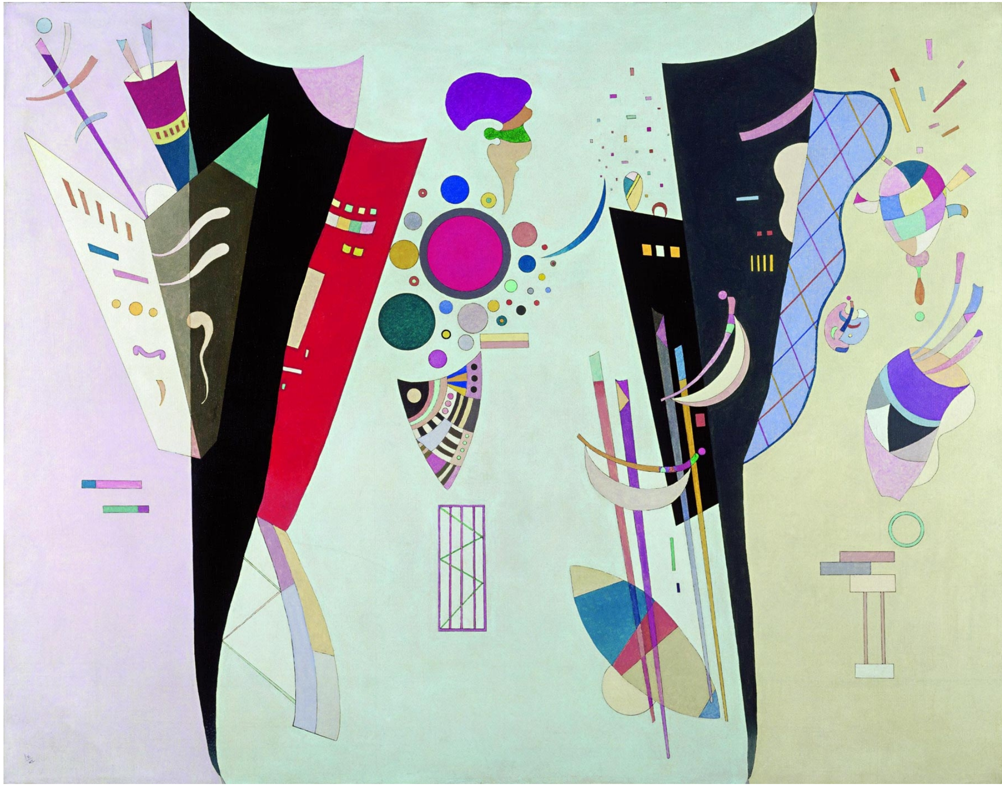
An “abstract” reciprocity map

Kandinsky, *Reciprocal Accords*



If you are planning to submit a final project, please let me know as soon as possible. These will be due Friday, March 19.

The class field axiom for idèle class groups

$k=\mathbb{Q} \qquad \bar{k}=\bar{\mathbb{Q}}$

$A=\bigcup_{K} C_K \qquad A_K = A^{\mathrm{Gal}(\bar{k}/K)} = C_K$

(for L/K finite Galois, $C_L^{\mathrm{Gal}(L/K)} = C_K$

Class field axiom $\Leftarrow$ First Inequality + Second Inequality

L/K cyclic

$$\#\hat{H}^i_T(\mathrm{Gal}(L/K), C_L) = \begin{cases} [L:K] & i \text{ even} \\ 1 & i \text{ odd.} \end{cases}$$

$$h(C_L) = [L:K] \qquad + \#\hat{H}^0_T \le [L:K]$$

Small cyclotomic extensions

$$\mathbb{Q}^{cyc} = \bigcup_n \mathbb{Q}(\zeta_n)$$

$$\mathrm{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^* = \prod_p \mathbb{Z}_p^*$$

$$(\mathbb{Z}_p^*)_{tors} = \begin{cases} \mathbb{Z}/(p-1)\mathbb{Z} & p>2 \\ \mathbb{Z}/2\mathbb{Z} & p=2 \end{cases}$$

$\mathbb{Z}_p^*/\text{torsion} \cong \mathbb{Z}_p$ via logarithms, but not canonically

$$\mathrm{Gal}(\mathbb{Q}^{smcy}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^*/\text{torsion} \cong \hat{\mathbb{Z}}$$

small cyclotomic extension — noncanonical but choose one.

"maximal unramified extension"

Abstract ramification theory

$$\begin{tikzcd} & \hat{\mathbb{Z}} \arrow[d, equal, "\cong\ \text{canonical}"] \\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \arrow[ur, "d"] \arrow[r] & \mathrm{Gal}(\mathbb{Q}^{smcy}/\mathbb{Q}) \end{tikzcd}$$

for L/K Galois

abstract inertia degree

$$f_{L/K} = [L \cap \mathbb{Q}^{smcy} : K \cap \mathbb{Q}^{smcy}]$$

abstract ramification index K^{unr}

$$e_{L/K} = [L:K] / f_{L/K}$$

$= K^{smcy}$

$= K\mathbb{Q}^{smcy}$

A candidate for the abstract henselian valuation

$$V : A_{\mathbb{Q}} = C_{\mathbb{Q}} \longrightarrow \widehat{\mathbb{Z}} \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{smcy}}/\mathbb{Q})$$

$$C_{\mathbb{Q}} = I_{\mathbb{Q}}/\mathbb{Q}^{*} \qquad \text{same noncanonical ism as picture.}$$

$$\cong \mathbb{R}^{+} \times \widehat{\mathbb{Z}}^{*} \longrightarrow \widehat{\mathbb{Z}}^{*} \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$$

need to check: ✓ $Z = \widehat{\mathbb{Z}}$

1. $V(C_{\mathbb{Q}})$ is a subgroup Z of $\widehat{\mathbb{Z}}$ containing $\mathbb{Z}$ with $Z/nZ \cong \widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}$ $\forall$ positive integers n.

2. $V(\mathrm{Norm}_{K/\mathbb{Q}}\, C_K) = f_{K/\mathbb{Q}}\, Z$

Verification of the abstract henselian valuation

Check $v(\mathrm{Norm}_{K/\mathbb{Q}}(x)) = f_{K/\mathbb{Q}}\hat{\mathbb{Z}}$

$\Uparrow$ Artin reciprocity for cyclotomic extensions

$$\mathbb{I}_K \xrightarrow{\mathrm{Norm}_{K/\mathbb{Q}}} \mathbb{I}_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^*$$

$$\mathrm{Gal}(K^{cyc}/K) \hookrightarrow$$

has image

(and likewise with cyc with smcy)

- For $K = \mathbb{Q}$, this follows from Artin reciprocity for cyclotomic extensions
- For general K, the same logic applies ~~if~~ if we can show that classical Artin map is surjective for $K(\zeta_n)/K$

A consequence of the First Inequality

Lemma For L/K abelian extension of # fields, classical Artin map is surjective.

Pf

if H = image of classical Artin map, $M = \mathrm{Fix}(H)$, then all but finitely many primes of K split completely in M.

we deduce from the First Inequality that: this is only possible if $K = M$.

The abstract reciprocity law

Thm For L/K Galois extension of # fields

$r_{L/K} : C_K / \mathrm{Norm}_{L/K} C_L \cong \mathrm{Gal}(L/K)^{ab}$

inherits some compatibilities with changing L & K

but not yet local-global compatibility.

$K = \mathbb{Q}$

$A = \bigcup_K C_K$

$d : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}$

$v : C_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$

satisfying class field axiom (1^{st} ≤ + 2^{nd} ≤)

and d/v compatibility

(Artin reciprocity for cyclotomic extensions)

The abstract norm limitation theorem

Thm $L|K$ ~~Galois~~ extension of # fields

$M|K$ abelianization $= L \cap K^{ab}$

Then $\mathrm{Norm}_{L|K} C_L = \mathrm{Norm}_{M|K} C_M$.

and both of these are subgroups

of finite index

A word on those artificial isomorphisms

Note: for $L \subset K^{sep}$ a finite cyclic extension,

then the map $C_K / \mathrm{Nm}_{L/K}\, C_L \xrightarrow{\sim} \mathrm{Gal}(L/K)$

matches "uniformizer" with "Frobenius"

the identities of these depend on the artificial isom $\mathrm{Gal}(k^{sep}/k) \cong \hat{\mathbb{Z}}$

but the map does not!

Preview: the approach to the adelic existence theorem

- For L/K Galois, $N_{L/K} C_L$ is an open subgroup of C_K (in the proof of second inequality)
- Conversely, every open subgroup of finite index in C_K is $N_{L/K} C_L$ for some finite (abelian) extension L/K Galois

Preview: reduction to the case of prime index

Enough to show: U contains $\mathrm{Norm}_{L/K}\, C_L$

for some L/K finite Galois (abelian)

then $\mathrm{Gal}(L/K) \cong C_K / \mathrm{Norm}_{L/K}\, C_L$

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \cong & C_K / \mathrm{Norm}_{L/K}\, C_L \\ \downarrow & & \downarrow \\ \mathrm{Gal}(M/K) & \cong & C_K / U. \end{array}$$

Preview: the use of Kummer extensions

Also, we assume $[C_K : U] = p$ prime

(otherwise induct: $U < V \subsetneq C_K$

Find $L|K$ with $N_{L|K} C_L \subseteq V$, pull back to L and argue there.)

Also, we reduce to case $\zeta_p \in K$

(b/c $[K(\zeta_p):K] \mid p-1$ coprime to p)

This case will be addressed using Kummer extension.