The existence theorem

https://xkcd.com/1856/
Statement of the theorem \( K = \mathbb{F}_{\ell} \)

Then the finite abelian extensions of \( K \) are in bijection with open subgroups of finite index \( \text{Gal}(K^a/K) \) via \( L \mapsto \text{Norm}_L \mathbb{F}_{\ell} \subseteq L^a \).
How this interacts with (abstract) reciprocity

From reciprocity, it follows that

- for $L/K$ abelian, $(\mathcal{C}_K : \text{Norm}_{L/K} \mathcal{C}_L) = \mathcal{C}_{L/K}$

and this super-

- For any given subgroup $U \subseteq \mathcal{C}_K$ open of finite index, if $U \supseteq \text{Norm}_{L/K} \mathcal{C}_L$ for some $L$,
  then $U = \text{Norm}_{L/K} \mathcal{C}_L$ for some $L$. 
Reduction to a key special case

This means we assume \( \mathfrak{c}(K^{1/n}) = p \), prime

and (assume \( K = K_p \)).

\[ K' = K(K_p), \quad U = \text{inversion map of } U\]

\( \mathfrak{c}(K', K') \downarrow_p \) corresponds to \( p \) and \( L' \), such that

\[ \text{Norm}_{L/K'} C_{L'} = U, \quad \rightarrow \quad \text{Norm}_{L/K} C_{L'} = U. \]
A special open subset $\mathcal{V}_K = \mathcal{V} \cdot \mathcal{U}_K$.

Lemma 7. A finite set of points $S$ contains all infinite planes, all plurigraus $S$, i.e., $-I_K = K^* I_K$ and preimage of $U$ in $I_K$ contains finiteness of $C^1(K)$, preimage of $pt$ at $U$ in $\mathcal{U}_K$.

$W_S = \prod (K^* V) \times \prod \mathcal{O}^*_K V V_S$ (open in $I_K$)
Local vs. global $p$-th powers

Lemma

For $w_s = \Pi_{v \in S} (K_v^*)^r \times \Pi_{v \in S} \mathcal{O}_{K_v}^*$, $w_s \cap K^*$

Proof

It follows that $P \subseteq (\mathcal{O}_{K}^*, s)$. Needs to show that $L = K(x^{1/p})$. By hypothesis, need to show $N_{M/K} c_L = c_K$.

If $w_s \cap K^*$ holds, then $w_s = W_s \cap K^*$.

Reference
A lemma on local $p$-th powers

For $p = p + 1$, place at $K$

\[
\left(\left(K_v^*: (K_v^*)^p \right) \right) = \frac{p^2}{\prod_{v \mid \mathfrak{p}} \mathfrak{p}}.
\]

Sketch of proof

Finite, pure to $p$,

$\prod_{v \mid \mathfrak{p}} \mathfrak{p}$

Conjugates $p$ factors of $p$ contribute nothing.

\[
K_v^* = \prod_{v \mid \mathfrak{p}} \mathfrak{p} \times \left(\mathfrak{p}^{\mathfrak{p} - 1}\right) \times U,
\]

Use $\lambda(v)$

\[
\sqrt{p}^v = p^2 \quad \text{and} \quad K_v^* \quad \text{divisible}
\]

Real $p = 2$

$\mathfrak{p}^\mathfrak{p} / (\mathfrak{p}^\mathfrak{p})^p = R^*/R^* = \mathbb{Z}/2\mathbb{Z}$
A candidate Kummer extension

\[ L = K \left( w^{1/p} : w \in \Omega_k^1 \right) \]

\[ (L : K) = p^s \]

\[ \mathbb{F}_p \subseteq \mathbb{F}_{p^s} \]

From local reciprocity, \( N_{O_K^1 / L} c_L = \mathbb{F}_{p}/\mathbb{F}_p \)

We'll show this equality!

Thus we show

\[ \left( \frac{K^*/\mathbb{F}_p, \mathbb{F}_{p^s}}{K^*/\mathbb{F}_p} \right) = \left( \mathcal{U}_K : N_{O_K^1 / L} \mathcal{U}_K \right) \]
Computation of the norm group of the candidate

\[ 1 \rightarrow O_{K^s}^* \rightarrow \frac{I_{K^s}}{\mathcal{O}_{K^s}^*} \rightarrow \frac{\mathcal{O}_{K^s}}{K^* W_{K^s}/K^s} \rightarrow 1 \]

\[ \prod \frac{r^2}{\sqrt{q_s 16}} = r^{2g} \]

\[ 1 \rightarrow \mathcal{O}_{K^s}^* \rightarrow \mathcal{I}_{K^s} \rightarrow \mathcal{C}_{K^s} \rightarrow 1 \]
Applying this idea to the Second Inequality

To reduce the second inequality to the case of a cyclic extension $K \subset K'$ of prime degree $p$, where $K = \mathbb{Q}$. 

$L = K(y^{\frac{1}{p}}), y \in K^*$

So $L$ is contained in $K(u^{\frac{1}{p}}: u \in \mathbb{Q}_{K^*})$ for some large enough finite set of places (say, containing all infinite places and places above $p$) such that $K \cap K_{s} = \mathbb{Q}$.
Presentation of a Kummer extension

Lemma For $s = \# S$, I can choose a second, disjoint set $T$ of $s-1$ finite places of $K$ that are both $\mathcal{O}^*$-invariant. This implies $O^*_K S = \prod_{\mathfrak{p} \in T} (K_{\mathfrak{p}}^*)^n$ for some positive integer $n$.

This lemma is essential for understanding the structure of the Kummer extension. In particular, it allows us to express the action of $O^*_K$ on $S$ in terms of the action of $K_{\mathfrak{p}}^*$ on $S$.

Proof Consider the ring $R$ of finite primes of $K$ that do not split into $\mathbb{N}_p$. Let $\mathfrak{p}$ be a prime such that $\mathfrak{p}$ does not split in $K$.
Local $p$-th powers revisited

In settings: \[ W_{s,t} = \left( \prod (K^*)^P \right) \times \prod (K^*) \times \prod \langle \mathcal{O}_K \rangle \]

\[ \ker \mathcal{K}^* \cap W_{s,t} = (\mathcal{O}_K^*)^P \]

for $m = K(y, \mathcal{O}_K)$

Let:

\[ \mathcal{N}_{\text{numm}/K} \mathcal{I}_{m, su} \neq \mathcal{I}_{K, su} \]

\[ \omega^+ \]

\[ \mathcal{N}_{\text{numm}/K} c_L = c_K \]
A bound on the norm subgroup

\[ \begin{aligned}
1 & \rightarrow \mathcal{O}_K \mathcal{S}_U \mathcal{T} \\
\mathcal{O}_K \mathcal{S}_U \mathcal{T} & \rightarrow I \mathcal{S}_U \\
I \mathcal{S}_U & \rightarrow C_K \\
C_K & \rightarrow \mathcal{K}/\mathcal{K}_1 \mathcal{K}^* \\
\end{aligned} \]
Next time after we prove local-global compatibility use similar strategy to prove: V prime at # Field # every abelian extension of K # is complex but at some abelian extension of K
(f or K # follows from # Frobenius - Liebew local K - W.)