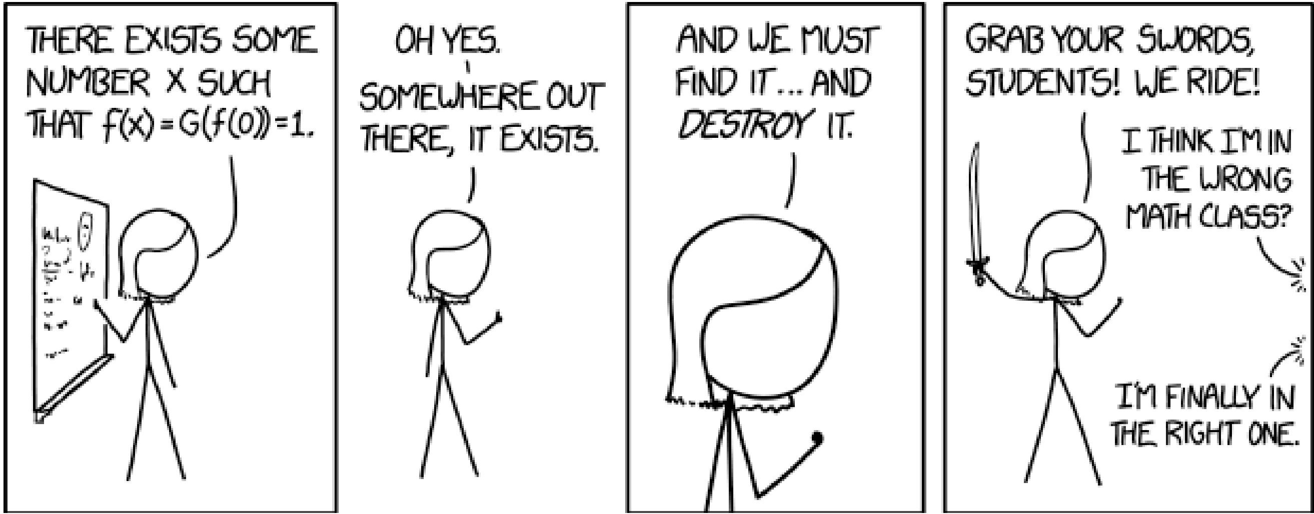
The existence theorem



https://xkcd.com/1856/

Statement of the theorem

$K = \#$ field

Thm The finite abelian extensions of K are in bijection with open subgroups of ~~fin~~ finite index in C_K via $L \longmapsto \mathrm{Norm}_{L/K}\, C_L$.

$\uparrow$ idele class group

How this interacts with (abstract) reciprocity

From reciprocity, it follows that

- for L/K abelian, $(C_K : \mathrm{Norm}_{L/K} C_L) = (L:K)$

  and this is open and quotient $\cong \mathrm{Gal}(L/K)$

- For any given subgroup $U \subseteq C_K$ open of finite index
  if $U \supseteq \mathrm{Norm}_{L/K} C_L$ for some L
  then $U = \mathrm{Norm}_{L/K} C_L$ for some L.

Reduction to a key special case

This means: can assume $[K:\mathbb{Q}] = p$ prime

and can assume $K \to \mathbb{Q}_p$.

$K' = K(\zeta_p)$, $U' =$ inverse image of U in $C_{K'}$

$[K':K] \neq p$ — coprime to p. Find L'/K' such that

$\mathrm{Norm}_{L'/K'} C_{L'} = U' \longrightarrow \mathrm{Norm}_{L'/K} C_{L'} = U.$

A special open subset

$(C_K : \underset{\text{open}}{U}) = p$, $\gamma_p \in K$

Lemma $\exists$ a finite set of places S containing all infinite places, all places above p s.t. $-I_K = K^* I_{K,S}$ and preimage of U in I_K contains

$$\boxed{W_S = \prod_{v \in S} (K_v^*)^p \times \prod_{v \notin S} \mathcal{O}_{K_v}^*} \quad \text{(open in } I_K\text{)}$$

finiteness of $Cl(K)$

preimage of pt U contains $\underbrace{\prod_{v \notin S} \mathcal{O}_{K_v}^*}$

and I_K^p

Local vs. global p-th powers

(a la Grunwald–Wang)

Lemma For $W_S \cap \mathcal{O}_{K,S}^* = (\mathcal{O}_{K,S}^*)^p$

$$W_S = \prod_{v \in S} (K_v^*)^p \times \prod_{v \notin S} \mathcal{O}_{K_v}^*, \quad = W_S \cap K^*$$

Pf $(\mathcal{O}_{K,S}^*)^p \subseteq W_S \cap \mathcal{O}_{K,S}^*$ clear.

Reverse: given $y \in W_S \cap \mathcal{O}_{K,S}^*$, need to show that $L = K(y^{\frac{1}{p}})$ equals K.

By first ineq, suffices to show $\to \mathrm{Norm}_{L/K} C_L = C_K$

I'll show $\mathrm{Norm}_{L/K} I_{L,S} = I_{K,S}$; for $v \in S$, L/K split at v; for $v \notin S$ L/K unramified at v.

box

$\square$

A lemma on local p-th powers

For p = prime, v place of K

$$[(K_v^*):(K_v^*)^p] = p^2/|p|_v.$$

product over v equals 1 by product formula.

sketch of proof: v finite, prime to p:

$K_v^* = \pi_v^{\mathbb{Z}} \times \langle \zeta_{q-1} \rangle \times U_1$, so $K_v^*/(K_v^*)^p = \langle \pi_{K_v}, \zeta_p \rangle$

π_v contributes 1 factor of p, ζ_{q-1} contributes 1 factor of p, U_1 contributes nothing.

q = # residue field

v complex. $1/|p|_v = p^2$ and K_v^* divisible

v real $\Rightarrow p=2$ $K_v^*/(K_v^*)^p = \mathbb{R}^*/\mathbb{R}^+ \cong \mathbb{Z}/2\mathbb{Z}$ / for $p>2$

$v|p$: $K_v^* = \pi_v^{\mathbb{Z}} \times \langle \zeta_{q-1} \rangle \times U_1$ ← use logarithm map; $\cong \mathbb{Z}_p^{n} \times \ldots$ $|p|_v = p^{-n}$.

A candidate Kummer extension

$s := \# S$

$$L = K\left(u^{1/p} : u \in \mathcal{O}^\times_{K,S}\right)$$

$$(L:K) = p^s$$

$\mathcal{O}^\times_{K,S} \cong \text{finite cyclic} \times \mathbb{Z}^{s-1}$

$$\# \mathcal{O}^\times_{K,S} / (\mathcal{O}^\times_{K,S})^p = p^s$$

From local reciprocity, $\mathrm{Norm}_{L/K}\, C_L \supseteq K^\times W_S / K^\times$

we'll show this is an equality!

so thus to check

$$\left[\frac{K^\times I_{L,S}}{K^\times} \cdot \frac{K^\times W_S}{K^\times}\right] = \left[C_K : \mathrm{Norm}_{L/K}\, C_K\right) \overset{\text{global reciprocity}}{=} [L:K) = p^s.$$

C_K

Computation of the norm group of the candidate

$$1 \to \frac{\mathcal{O}_{K,S}^{\times}}{\cancel{\mathcal{O}_{k,S}^{\times}\,\mathcal{O}_{K,S}^{\times p}}} \to \frac{I_{K,S}}{W_S} \to \frac{C_K}{K^\times W_S / K^\times} \to 1$$

$$= \underbrace{\left(\mathcal{O}_{k,S}^{\times}\right)^p}_{p^s}$$

$$\prod_{v \in S} \frac{p^2}{|p|_v} = p^{2s}$$

$$\frac{C_K}{K^\times W_S / K^\times} : p^s$$

$$\square$$

$$1 \to \mathcal{O}_{K,S}^{\times} \to I_{K,S} \to C_K \cdot$$

Applying this idea to the Second Inequality

Can reduce the Second Inequality
to the case of a cyclic extension L/K
of prime degree p where $K \ni \zeta_p$.

$L = K(y^{1/p})$ $y \in K^*$

So L is contained in $K(u^{1/p} : u \in \mathcal{O}_{K,S}^*) = N$
for some large enough finite set S of
places (say, containing infinite places
& places above p)
so that $K^* I_{K,S} = I_K$

Presentation of a Kummer extension

<u>Lemma</u> For $s = \# S$, I can choose a second, disjoint set T of $s-1$ finite places of K

s.t. $\mathcal{O}_{K,S}^* \to \prod_{u \in T} (K_u^*)/(K_u^*)^p$ is surjective

and its kernel Δ generates L as a Kummer ext

<u>pf</u> Choose Cebotarev inequality to generate primes of K that do not split from N_i to N

write $L = N_1 \cap \dots \cap N_{s-1}$ some index-p subfields,

Local p-th powers revisited

In setting: $W_{S,T} = \prod_{v \in S} (K_v^*)^p \times \prod_{v \in T} K_v^* \times \prod_{v \notin S \cup T} \mathcal{O}_{K_v}^*$

then $K^* \cap W_{S,T} = (\mathcal{O}_{K, S \cup T}^*)^p$ for $M = K(y^{1/p})$, $y \in K^* \cap W_{S,T}$

Idea: $\mathrm{Norm}_{M/K}\, I_{M,S \cup T} \neq I_{K,S \cup T}$

but

$\mathrm{Norm}_{M/K}\, C_L = C_K$

A bound on the norm subgroup

$$1 \longrightarrow \underbrace{\frac{O_{K,S\cup T}^{*}}{\cancel{O_{K,S\cup T}^{*} \cap W_{S,T}} \quad (O_{K,S\cup T}^{*})^{p}}}_{\text{order } p^{2s-1}} \longrightarrow \underbrace{\frac{I_{K,S\cup T}}{W_{S,T}}}_{p^{2s}} \longrightarrow \underbrace{\frac{C_K}{\cancel{K^*} W_{S,T} K^{*}}}_{p} \longrightarrow 1$$

Preview: an application of local-global compatibility

Next time after we prove local-global compatibility, use similar strategy to prove: $\forall$ place of # field K, every abelian extension of K_v is completion of some abelian extension of K.

(for $K=\mathbb{Q}$, follows from Kronecker-Weber + local K-W.)