

Local-global compatibility

Last week of lectures!

Reminder: I will not be teaching Math 204C, but I do plan to keep the Zulip running during the spring term for those who want to follow the lecture notes. My course next term (Math 206: a topics course on prismatic cohomology) will have a separate Zulip.



Statement of compatibility

L/K Galois extension of \mathbb{A}^1 fields.

We want to equate:

$$r_{L/K}: \mathbb{I}_K \longrightarrow \text{Gal}(L/K)^{ab}$$

product of local reciprocity maps

$$r_{L/K}(\alpha v) = \prod_v r_{L_w/K_v}(\alpha v)$$

$$\text{Gal}(L_w/K_v)^{ab} \rightarrow \text{Gal}(L/K)^{ab}$$

also has norm when v is infinite

In particular we need:
 $r_{L/K}(L^*) = \{e\}$.

w/ M

$$r'_{L/K}: \mathbb{I}_K \longrightarrow \text{Gal}(L/K) \quad \text{defined using construct } \mathbb{C}^T,$$

$$\downarrow \quad \uparrow$$

$$\mathbb{C}_K \longrightarrow \mathbb{C}_K \# \text{Norm}_{L/K} \mathbb{C}_L$$

The cyclotomic case

For $L \subseteq K(\zeta_n)$ we can read this off directly.

$$\text{Frob}_p \quad \text{Gal}(L/K) \subseteq (\mathbb{Z}/n\mathbb{Z})^*$$

Fix $f \neq n$.

by direct calculation, Artin reciprocity holds and $\text{Gal}(K^*/K) = \{e\}$.

$\zeta = \zeta_n$ is a primitive n th root of unity

Now $\text{Gal}(L/K)$ is defined directly in terms of Artin map

when $L \subseteq K^{\text{sm}}$ "L/K unramified"

"L/K sends 'uniformizer' to 'Frobenius'"

$$\text{Frob}_p \in \text{Gal}(K^{\text{sm}}/K) \mapsto 1 \pmod{n}$$

$$\text{Gal}(K^{\text{sm}}/K) \xrightarrow{d} \mathbb{Z}/n\mathbb{Z}$$

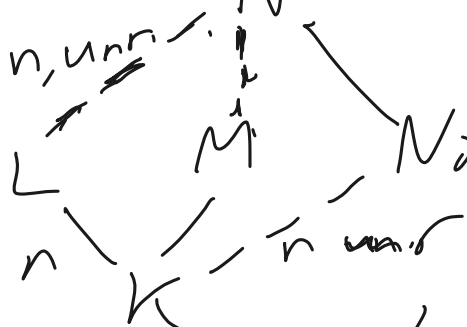
A remark about the cyclotomic case

It is also possible to explicitly compute all the local reciprocity maps for a cyclotomic extension via Lubin-Tate theory, and then check directly that $v_{L/K}(K^*) = \{e\}$.

Compatibility for "totally ramified" extensions

Suppose L/K Galois and $L \cap K^{\text{sm}} = K$

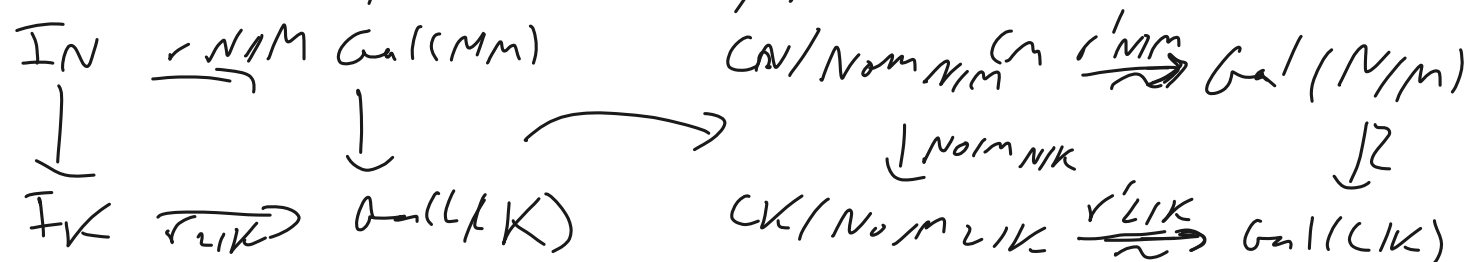
Recall we've shown in this case that $r_{L/K}^{\text{sm}}$ is normal. $\frac{GK}{\text{Norm}_{L/K} G_L}$



Claim: reduce equality

$$r_{L/K} = r'_{L/K} \text{ from}$$

L/K to N/M



Compatibility for general extensions

For E/K general Galois extension,
replace L with K^{ab} to reduce to L/K abelian.

$$1 \rightarrow \text{Gal}(K^{ab}/K) \xrightarrow{\text{incl}} \text{Gal}(K^{ab}/K) \rightarrow \text{Gal}(K^{sm}/K) \rightarrow 1$$

choose a splitting of exact sequence.

$\Rightarrow L$ is unramified in L_1, L_2 L_1/K abelian
"totally unramified"
 L_2/K abelian
"unramified"

Look at these two cases separately.

$$\delta_{L/K} \geq v_{L/K}$$

\Rightarrow deduce $v_{L/K}(K^*) = \{0\} \Rightarrow$ Artin reciprocity. Mordell! □

A remark on higher reciprocity laws

The fact that $r_{L/K}(K^{\times}) = \langle e \rangle$
is the basis of higher reciprocity laws
(Gauss, Hilbert)

Globalization of local abelian extensions

Thm $K = \mathbb{A}$ field, $v = \mathfrak{p} \mid n$ of K , M/K_v
finite abelian.

then $\exists L/K$ finite abelian, s.t. for any place w
of L above v , $M \subseteq L_w$.

PF $v = \infty$ finite, this case is easy. $L = K(\sqrt[n]{\cdot})$

$v = \text{finite}$:

local-global compatibility: it's sufficient to find $\sqrt[n]{\cdot} \subseteq K$
open of K finite index
& existence theorem s.t. $K \sqrt[n]{\cdot} \cap V$ is contained in
 $N = \text{Norm}_{M/K_v}$.

Globalization of local abelian extensions: proof

$S = \{ \text{finite places of } K \}$

$T = S \cup \{ \infty \}$

$\mathcal{O}_{K, T}^* = \text{Kronecker ab group}$

$G = \mathcal{O}_{K, T}^* \cap N = \text{Kronecker index subgroup}$

Pick a place $u \notin T$. Image of $\mathcal{O}_{K, T}^*$ in K_u^* is finite, so \exists neighborhood \mathcal{U} of $1 \in K_u^*$ s.t. $\mathcal{U} \cap \mathcal{O}_{K, T}^* \subseteq G$.

Put $W = N \times \mathcal{U} \times \prod_{w \in S} K_w^* \times \prod_{w \notin T \cup S} \mathcal{O}_{K_w}^*$
 $\subseteq \mathbb{I}_K$ open, finite index

Preview: the Brauer group of a field

Claim $K^*W/K^* \subset CK$ w/m.s.
 $= V$

If $\alpha \in K_{\sqrt{}}^*$ induces V , then

- $\exists \beta \in K^* \quad \alpha \beta \in W$

- $\alpha \beta \in N$

- $\beta \in \mathcal{O}_{K, \mathfrak{p}}^* \quad \beta \in \mathcal{U} \Rightarrow \beta \in \mathcal{U}$
 $\Rightarrow \beta \in N.$

$\Rightarrow \alpha \in N. \quad \checkmark$

Preview: an approach to compatibility via Brauer groups

An alternate approach to proving that

$$\text{Gal}(K) \text{ acts on } (K^*) = \langle e \rangle$$

is to use $H^2(\text{Gal}(\bar{K}/K), \bar{K}^*)$

= Brauer group of K

Fundamental exact sequence

$$1 \rightarrow B_r(K) \rightarrow \textcircled{1} B_r(K_v) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 1$$

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