Local-global compatibility

Last week of lectures!

Reminder: I will not be teaching Math 204C, but I do plan to keep the Zulip running during the spring term for those who want to follow the lecture notes. My course next term (Math 206: a topics course on prismatic cohomology) will have a separate Zulip.

Statement of compatibility

L/K Galois extension of # fields.

We want to equate:

$$r_{L/K} : \mathbb{I}_K \longrightarrow \mathrm{Gal}(L/K)^{ab}$$

product of local reciprocity maps

$$r_{L/K}((\alpha_v)_v) = \prod_v r_{L_w/K_v}(\alpha_v)$$

also has meaning when v is infinite

$$\mathrm{Gal}(L_w/K_v)^{ab} \to \mathrm{Gal}(L/K)^{ab}$$

In particular we need: $r_{L/K}(K^*) = \{e\}$.

with

$r'_{L/K} : \mathbb{I}_K \longrightarrow \mathrm{Gal}(L/K)$ defined using abstract CFT,

$$C_K \longrightarrow C_K / \mathrm{Nm}_{L/K}\, C_L$$

The cyclotomic case

For $L \subseteq K(\zeta_n)$ we can read this off directly.

$$\mathrm{Frob}_p \longmapsto q \qquad \mathrm{Gal}(L/K) \subseteq (\mathbb{Z}/n\mathbb{Z})^*$$

q = order of residue field at p

Fix $p \nmid n$.

by direct calculation, Artin reciprocity holds and $r_{L/K}(K^*) = \{e\}$

Now $r_{L/K}$ is defined directly in terms of Artin map when $L \subset K^{\mathrm{ur}}$ "L/K unramified"

$r_{L/K}$ sends "uniformizer" to "Frobenius"

$\pi_K \in \mathcal{O}_K \quad v(\pi_K) = 1 \qquad \mathrm{Gal}(K^{\mathrm{ur}}/K) \xrightarrow{\sim} \hat{\mathbb{Z}}, \; \mathrm{Frob} \mapsto 1$

A remark about the cyclotomic case

It is also possible to explicitly compute all the local reciprocity maps for a cyclotomic ext'n via Lubin-Tate theory, and then check directly that $r_{L/K}(K^*) = \{e\}$.

Compatibility for "totally ramified" extensions

Suppose L/K abelian and $L \cap K^{\mathrm{unr}} = K$

Recall how we showed in this case that $r'_{L/K}: \dfrac{C_K}{N_{L/K} C_L} \xrightarrow{\sim} \mathrm{Gal}(L/K)^{ab}$ is an isom.

$$\begin{tikzcd}
 & & N \arrow[dll, dash, dashed, "n\text{, unr.}"'] \arrow[d, dash, dashed] \arrow[dr, dash] & \\
L \arrow[drr, dash, "n"'] & & M \arrow[d, dash] & N_0 \arrow[dl, dash, "n\text{, unr}"] \\
 & & K &
\end{tikzcd}$$

Claim: reduce equality $r_{L/K} = r'_{L/K}$ from L/K to N/M

$$\begin{tikzcd}
I_N \arrow[r, "r_{N/M}"] \arrow[d] & \mathrm{Gal}(N/M) \arrow[d] \\
I_K \arrow[r, "r_{L/K}"] & \mathrm{Gal}(L/K)
\end{tikzcd}
\qquad \longrightarrow \qquad
\begin{tikzcd}
C_N/\mathrm{Nm}_{N/M} C_M \arrow[r, "r'_{N/M}", "\sim"'] \arrow[d, "\mathrm{Norm}_{N/K}"] & \mathrm{Gal}(N/M) \arrow[d, "\wr"] \\
C_K/\mathrm{Nm}_{L/K} C_L \arrow[r, "r'_{L/K}", "\sim"'] & \mathrm{Gal}(L/K)
\end{tikzcd}$$

Compatibility for general extensions

For E/K general Galois extension,

replace L with K^{ab} to reduce to L/K abelian.

$$1 \to \mathrm{Gal}(K^{ab}/K^{smly}) \to \mathrm{Gal}(K^{ab}/K) \to \underbrace{\mathrm{Gal}(K^{smly}/K)}_{\simeq \hat{\mathbb{Z}}} \to 1$$

choose a splitting of exact sequence.

$\Rightarrow$ L is contained in $L_1 L_2$ $\quad$ L_1/K abelian "totally ramified"

L_2/K abelian "unramified"

Look at these two cases separately. $\square$

$r_{L/K} \Rightarrow r_{E/K}$

$\Rightarrow$ deduce $r_{L/K}(K^*) = \{e\}$ $\Rightarrow$ Artin reciprocity. Hooray!

A remark on higher reciprocity laws

The fact that $r_{L/K}(K^*) = \{e\}$

is the basis of higher reciprocity laws

(Gauss, Hilbert)

Globalization of local abelian extensions

Thm K = # field, v = place of K, M/K_v finite abelian.

then $\exists L/K$ finite abelian, s.t. for any place w of L above v, $M \subseteq L_w$.

Pf v = infinite, this case is easy. $L = K(\sqrt{-1})$

v = finite:

Local-global compatibility + existence theorem: it suffices to find $V \subseteq C_K$ open of finite index s.t. $K_v^\times \cap V$ is contained in $N = \mathrm{Norm}_{M/K_v} M^\times$.

Globalization of local abelian extensions: proof

$S = \{$infinite places of $K\}$

$T = S \cup \{v\}$ $\qquad \mathcal{O}^*_{K,T}$ = fin gen ab group

$G = \mathcal{O}^*_{K,T} \cap N$ = finite index subgroup of $\mathcal{O}^*_{K,T}$.

Pick a place $u \notin T$. Image of $\mathcal{O}^*_{K,T}$ in K_u^* is finite, so $\exists$ neighborhood U of $\{e\}$ in K_u^* s.t. $U \cap \mathcal{O}^*_{K,T} \subseteq G$.

Put $W = \underset{v}{N} \times \underset{u}{U} \times \prod_{w \in S} K_w^* \times \prod_{w \notin T \cup \{u\}} \mathcal{O}^*_{K_w}$

$\subseteq I_K$ open, finite index

Preview: the Brauer group of a field

Claim $K^*W/K^* \subset C_K$ nbhs.

$= V$

If $\alpha_v \in K_v^*$ lands in V, then

- $\exists\ \beta \in K^*$ $\alpha_v \beta \in W$
- $\alpha_v \beta_v \in N$
- $\beta \in \mathcal{O}_{K,T}^*$ and $\beta_u \in U \Rightarrow \beta \in G$

  $\Rightarrow \beta_v \in N$.

$\Rightarrow \alpha_v \in N$. ✓

Preview: an approach to compatibility via Brauer groups

An alternate approach to proving that

$$r_{L/K}(K^*) = \{e\}$$

is to use $H^2(\mathrm{Gal}(\bar{K}/K), \bar{K}^*)$

$= \underline{\text{Brauer group of } K}$

<u>Fundamental exact sequence</u>

$$1 \to Br(K) \to \bigoplus_v Br(K_v) \to \mathbb{Q}/\mathbb{Z} \to 1$$