Brauer groups and the reciprocity map

Last lecture on CFT.

Next time: preview of 204C

(Tate's thesis)

Richard Brauer (AMS President, 1957-58)

The Brauer group of a field: cohomological definition

Let K be a field

The Brauer group of K is

$$Br(K) = H^2(Gal(\bar{K}/K), \bar{K}^*)$$

$$= \varinjlim_{L/K \text{ finite Galois}} H^2(L/K, L^*)$$

Reminder: $H^1(L/K, L^*) = \{1\}$ by Thm 90,

so Inflation-Restriction $\Rightarrow$ direct limit is a union.

The Brauer group of a field: classical definition

Lemma Let K be a field. Consider the set of isomorphism classes of division algebras with center K (skew fields) which are finite-dim^{e} over K.

- There is a bijection of sets with $Br(K)$
- The resulting group operation is:

$$D_1 \times D_2 = D \quad \text{where} \quad D_1 \otimes_K D_2 = M_n(D)$$

- identity is K
- inverse of D is D^{op} = opposite algebra

Examples - K = algebraically closed $Br(K) = \{1\}$

- K finite, $Br(K) = \{1\}$
  (Wedderburn's theorem)

  in cohomological interpretation, this follows from
  exercise: if L/K ext. of finite fields, then

  $\mathrm{Norm}_{L/K} L^* = K^*$, so $0 = H^0_T(L/K) = H^2_{(T)}(L/K)$

  $\underline{\text{cyclic}}$

- $K = \mathbb{R}$ $Br(K) = \mathbb{Z}/2\mathbb{Z}$ (classes of $\mathbb{R}$, $\mathbb{H}$)

  $H^2(\mathbb{C}/\mathbb{R}) = H^0_T(\mathbb{C}/\mathbb{R}) = \mathbb{R}^*/\mathrm{Norm}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^*$

  $= \mathbb{R}^*/\mathbb{R}^+ = \{\pm 1\}$.

Remark: Brauer groups in representation theory

K = field of char 0

if $Br(K) = \{1\}$, then for every finite group G, every K-valued irreducible character of G is realized by a K-linear irreducible representation.

e.g. $G = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$ $\quad \chi(e) = 2.$

has a $\mathbb{Q}$-valued character of dim 2

not realizable as $\mathbb{Q}$-linear representation.

__Note__: importantly, $Br(K) = \{1\}$ when $K = \mathbb{Q}^{cyc}$

Remark: Brauer groups of more general objects

One can also define Brauer groups of rings and even schemes.

- analogue of central simple algebra is Azumaya algebra

- analogue of Galois cohomology is étale cohomology — but the analogy is not perfect.

The Brauer group of a number field: statement

Thm For any number field K there is ~~an~~ exact sequence i.e.

Fundamental exact sequence of CFT

$$1 \to \mathrm{Br}(K) \to \bigoplus_{v \text{ places}} \mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 1$$

in which

$$\mathrm{Br}(K_v) = \begin{cases} \mathbb{Q}/\mathbb{Z} & v \text{ finite (local reciprocity)} \\ \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} & v \text{ real} \\ 0 & v \text{ complex} \end{cases}$$

with distinguished

The Brauer group of a number field: proof (part 1)

$Br(K) \longrightarrow \bigoplus_v Br(K_v)$ injective [Albert-Brauer-Hasse-Noether]

values of $Br(K_v)$ come from local reciprocity

(reminder: $H^2(\overline{K_v}/K_v) = H^2(K_v^{unr}/K_v)$

is $\mathbb{Q}/\mathbb{Z}$ with a distinguished generator

A commutative diagram

For L/K cyclic extension of # fields, $n = [L:K]$

$$\begin{tikzcd}
K^*/\mathrm{Norm}_{L/K} L^* = H_T^0(\mathrm{Gal}(L/K), L^*) \arrow[r] \arrow[d, "\wr"] & I_K/\mathrm{Norm}_{L/K} I_L = H_T^0(\mathrm{Gal}(L/K), I_L) \arrow[r, "\mathrm{rec}_{L/K}", "\sim"'] \arrow[d, "\wr"] & \mathrm{Gal}(L/K) \arrow[d, "\wr"] \\
H^2(\mathrm{Gal}(L/K), L^*) \arrow[r] & H^2(\mathrm{Gal}(L/K), I_L) \arrow[r] & \frac{1}{n}\mathbb{Z}/\mathbb{Z}
\end{tikzcd}$$

is a commutative diagram.

Left square: periodicity of Tate groups

Right square local reciprocity.

The Brauer group of a number field: proof (part 2)

For L/K Galois ext of # fields, I have an exact sequence

$H^2(\mathrm{Gal}(L/K), L^\times) \to H^2(\mathrm{Gal}(L/K), I_L) \to H^2(\mathrm{Gal}(L/K), C_L)$

if L/K cyclic this is top row of the previous diagram so by the snake lemma exact.

It would now suffice to see that every class in $\mathrm{Br}(K)$ is image of some class in $H^2(L/K)$ for some cyclic extension L/K.

Lemma: all Brauer classes are cyclotomic

For L/K Galois ext. of # fields, $x \in H^2(L/K)$

$\exists$ cyclic cyclotomic extension M/K

and a class $y \in H^2(M/K)$

s.t. x, y have same image in $H^2(ML/K)$

Pf. By Albert-Brauer-Hasse-Noether, suffices to account for x locally.

- follows from local reciprocity (exercise).

Local-global compatibility via Brauer groups

Can use Brauer groups to recover
local-global compatibility for L/K
without comparison with $r_{L/K}$!

- For L/K cyclotomic, use explicit Artin reciprocity.

to show that $r_{L/K}(K^\times) = 1$.

Local-global compatibility via Brauer groups

For L/K cyclic cyclotomic, we must have that

$$H^2(L/K) \longrightarrow \bigoplus H^2(L_v/K_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is zero. By [illegible] lemma, same is true for any L/K cyclic, then use dim. arg. from earlier to reduce $r_{L/K}(K^*) = \{1\}$.

The non-abstract approach to class field theory

Also see that for L/K cyclic

$$r_{L/K}: C_K / Nm_{L/K} C_L \to C(L/K)$$

is well defined, and surjective (from point of 1st inequality), and source & target have same order (by 1st+2nd inequality)!

$\Rightarrow r_{L/K}$ is an isomorphism

Also gives $r_{L/K}$ isom for any L/K abelian without abstract CFT. However, does not immediately give norm limitation or existence theorem.