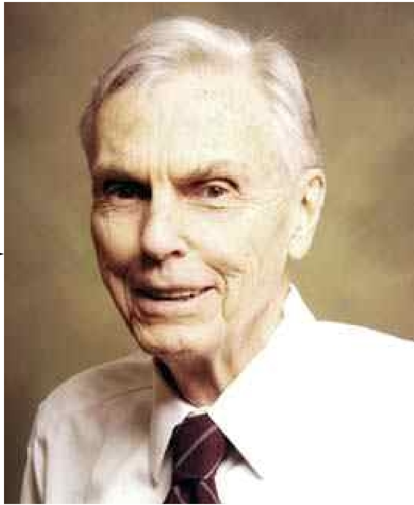
Adelic Fourier analysis: preview of Math 204C

Last lecture! Thank you for attending.

Although I am not teaching Math 204C, I will keep the Zulip open for discussion throughout the spring. I do plan to hold some office hours for epicourse participants; watch Zulip for announcements.

As of today, the Math 204C home page is not yet available. I will post the link to Zulip, and the 204B web site, once I have it.



Joseph Fourier and John Tate (from MacTutor History of Mathematics Archive)

Additive characters: local case

K = # field

v = place of K.

K_v^+ = additive group of K_v as a locally compact topological group.

$(K_v^+)^\vee$ = dual group = group of characters of K_v^+

= continuous homomorphisms $K_v^+ \to \{z \in \mathbb{C} : |z| = 1\}$

Lemma $(K_v^+)^\vee$ is again locally compact top group

In fact for any single nonzero $\chi \in (K_v^+)^\vee$, get isomorphism

$$K_v^+ \xrightarrow{\sim} (K_v^+)^\vee \qquad \eta \longmapsto (\xi \to \chi(\eta\xi))$$

Explicit local additive characters

For $K = \mathbb{Q}$ $\quad v = \infty$: $\quad \chi : t \longmapsto e^{-2\pi i t}$

$v = p$: $\quad \chi : t \longmapsto e^{-2\pi i \lambda(t)}$

$\lambda(t) \in \mathbb{Z}[\frac{1}{p}]$ congruent to t mod $\mathbb{Z}_p$.

$\left(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \cong \mathbb{Q}_p/\mathbb{Z}_p \right)$

For general K, take trace to $\mathbb{Q}$ and use the same χ.

<u>Note:</u> χ is trivial on $\mathbb{Z}_p$

(or more generally, on $\mathcal{O}_{K_v}$)

Additive characters: global case

K = # field

For each place v of K (finite or infinite) define additive character $\chi_v \in (K_v^+)^\vee$ as above.

Put $\chi \in \mathbb{A}_K^{+\vee}$ $\quad \chi(\alpha) = \prod \chi_v(\alpha_v)$

finite product (b/c of normalization)

Thm Again, get an isomorphism

$$\mathbb{A}_K^+ \to \mathbb{A}_K^{+\vee} \quad \text{via}$$

$$\alpha \longmapsto (\beta \mapsto \chi(\alpha\beta))$$

Fourier inversion: local case

Fix a place v and Haar measure on K_v^+

(for v finite $\mu(\mathcal{O}_{K_v})=1$)

$f \in L^1(K_v^+)$

Define Fourier transform $\hat{f}(\eta) = \int f(\xi)\chi(\eta\xi)\,d\xi$

If $\hat{f} \in L^1(K_v^+)$,

$$f(\xi) = c\int \hat{f}(\eta)\chi(-\eta\xi)\,d\eta = c\,\hat{\hat{f}}(-\xi)$$

for some $c>0$ depending on Haar measure and χ.

In particular, can normalize things to force $c=1$

Fourier inversion: global case

$K = \#$ field

fix Haar measure on $\mathbb{A}_K$

$\chi \in (\mathbb{A}_K^+)^\vee$

$f \in L_1(\mathbb{A}_K^+)$

$$\hat{f}(\eta) = \int f(\xi)\chi(\eta\xi)\,d\xi$$

if $\hat{f} \in L_1(\mathbb{A}_K^+)$, then

$$f(\xi) = c\int \hat{f}(\eta)\chi(-\eta\xi)\,d\eta = c\hat{\hat{f}}(-\xi)$$

for some $c>0$ depending on Haar measure;

can normalize to force $c=1$.

The adelic Poisson summation formula

Classical PSF: use fact that $\mathbb{Z} \subset \mathbb{R}$ is discrete, cocompact in $\mathbb{R}$

to relate $\sum_{x \in \mathbb{Z}} f(x)$ to $\sum_{x \in \mathbb{Z}} \hat{f}(x)$

Adelic setup: use fact that $K \subset \mathbb{A}_K$ is discrete, cocompact

to relate $\sum_{x \in K} f(x)$ to $\sum_{x \in K} \hat{f}(x)$.

Quasi-characters on the idèle class group

K = # field

C_K = idèle class group

<u>quasicharacter</u> on C_K = continuous homomorphism $C_K \to \mathbb{C}^*$

(whereas <u>character</u> maps into $\{|z|=1\}$.)

$\mathbb{C}^* = \{|z|=1\} \times \mathbb{R}^+$

For each quasicharacter c on C_K, $\exists$ unique real number s s.t.

$$|c(\alpha)| = |\alpha|^s = \prod |\alpha_v|_v^s \quad \forall \alpha \in I_K$$

= <u>exponent</u> of c

The space of quasi-characters

The (group) space of quasi-characters contains a copy of

$$\mathbb{C} : s \longmapsto c_s : \alpha \to |\alpha|^s$$

exponent of this is $\mathrm{Re}(s)$

In adelic world the domain of the zeta function of K is space of quasicharacters!

This copy of $\mathbb{C}$ will correspond to usual Dedekind ζ;

other translates will correspond to Hecke L-functions

Paradigm shift: the zeta function on quasi-characters

Classically: interpret individual zeta function or L-function

as integral transform (Mellin) of some "automorphic" object.

Adelic: similarly, but for all L functions at once! Hecke

Definition of the zeta function

Start with a "test function" $f: \mathbb{A}_K \to \mathbb{C}$

Define, for c a quasi-character of C_K (extend by 0)

$$\zeta(f, c) = \int f(\alpha)\, c(\alpha)\, d\alpha$$

if exponent$(c) > 1$.

Analytic continuation of the zeta function

Thm This function extends by "analytic continuation" to a function on the entire space of quasicharacters except for "poles" at $s = 0$, $s = 1$

Residues at the poles

$$\text{Res at } s=0 = -K f(0)$$

$$\text{Res at } s=1 = K \hat{f}(0)$$

class #

unit regulator

where $K = \dfrac{2^{r_1}(2\pi)^{r_2}\, h R}{\sqrt{|\Delta_K|}\ \#\mu_K}$

real places

complex places

absolute discriminant

roots of unity

The functional equation for the zeta function

Moreover, we get a functional equation

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c})$$

$$\hat{c}(\alpha) = \alpha\, c(\alpha)^{-1}$$

$$(s, \quad \text{———} \quad (1-s)$$

About the test function

We need to take the test function so that

$f, \hat{f}$ are closely related.

For further details, see:

Cassels-Fröhlich, chapter XV

(original source: Tate's thesis)

Math 204C: Ramakrishnan-Valenza

Fourier Analysis on Number Fields