Math 204B (Number Theory), UCSD, winter 2021
Problem Set 10 – due Thursday, January 21, 2021

1. Let G be a finite group and let H be its commutator subgroup. Assuming the existence of the transfer homomorphism $V : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ as defined in CFT Theorem 2.3.7, prove that it is zero in this case. (Hint: see CFT exercise 2.3.6.5.)

2. Find the ray class field of $\mathbb{Q}(i)$ of conductor (3) and verify Artin reciprocity explicitly in this case.

3. Let D be an odd squarefree positive integer greater than 3. Give a formula for the order of the generalized ideal class group $\mathrm{Cl}^{\mathfrak{m}}(\mathbb{Q}(\sqrt{-D}))$ in terms of the class number of $\mathbb{Q}(\sqrt{-D})$ and the factors of D and the modulus $\mathfrak{m}$.

4. Here is an example to illustrate the difference between Dirichlet density and natural density, albeit not for primes. Let S be the set of positive integers whose decimal expansion begins with 1.

   (a) Prove that S does not have a natural density, in the sense that

   $$\lim_{X\to\infty} \frac{1}{X}\#(S \cap \{1, \ldots, X\})$$

   does not exist.

   (b) On the other hand, prove that S has a Dirichlet density in the sense that

   $$\lim_{s\to 1^+} \frac{\sum_{n\in S} n^{-s}}{\sum_{n=1}^{\infty} n^{-s}}$$

   exists, and compute this value.

5. Just for fun, here is another result due to Chebotarëv. Let p be a prime.

   (a) Let $f \in \mathbb{F}_p[x]$ be a nonzero polynomial of degree less than p. Prove that the multiplicity of any nonzero element of $\mathbb{F}_p$ as a root of f is less than the number of nonzero coefficients of f. (Hint: compare f and f'.)

   (b) Let M be the $p \times p$ matrix given by

   $$M = (\zeta_p^{ij})_{i,j=0}^{p-1}.$$

   Prove that every minor of the matrix M is nonzero. (Hint: reduce modulo $1 - \zeta_p$ and apply (a).)