Math 204B (Number Theory), UCSD, winter 2021
Problem Set 11 – due Thursday, January 28, 2021

1. Let $H \subseteq G$ be an inclusion of finite groups. Let I_G be the *augmentation ideal* of the group algebra $\mathbb{Z}[G]$, i.e., the kernel of the ring homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$ taking $\sum_g n_g[g]$ to $\sum_g n_g$; similarly, let I_H be the augmentation ideal of $\mathbb{Z}[H]$. Let G' and H' be the commutator subgroups of G and H. Prove that the map

$$\delta : H/H' \to (I_H + I_G I_H)/I_G I_H$$

taking h to the class of $[h] - 1$ is an isomorphism of abelian groups.

2. Let $\mathfrak{m}$ be a formal product of places of the number field K. Prove that for each element of the generalized ideal class group of K of conductor $\mathfrak{m}$, the set of primes of K whose Artin symbol (for the ray class field of conductor $\mathfrak{m}$) lies in that class has Dirichlet density $1/\#\operatorname{Cl}^{\mathfrak{m}}(K)$. (Hint: see CFT exercise 2.4.5.3.)

3. Let G be the trivial group. Prove that a G-module is injective if and only if it is divisible, i.e., the map $x \mapsto nx$ is surjective for any nonzero integer n. (Hint: you'll need Zorn's lemma or equivalent in one direction.)

4. Let G be a finite group. Let $0 \to M \to M^0 \to M^1 \to \cdots$ be an exact sequence of G-modules such that $H^i(G, M^j) = 0$ for all $i > 0, j \geq 0$. Prove that the cohomology groups of the complex $0 \to M^{0G} \to M^{1G} \to \cdots$ coincide with the groups $H^i(G, M)$. (Hint: separate the original sequence into two exact sequences

$$0 \to M \to M^0 \to M^0/M \to 0$$

and

$$0 \to M^0/M \to M^1 \to M^2 \to \cdots$$

and use the long exact sequence on the latter to set up an induction on i. Do not forget about the base case!)

5. Let G be the group S_3 (the symmetric group on three letters). Let M be the abelian group $\mathbb{Z}^3$ with the G-action given by permuting the coordinates. Let N be the subgroup M^G of M, viewed as a G-module via the trivial action. Compute $H^i(G, M/N)$ for $i = 1, 2$. (Hint: there are lots of ways to do this. One is to compute directly from the definition. Another is to first consider the subgroup $H = A_3$ of S_3.)