Math 204B (Number Theory), UCSD, winter 2021
Problem Set 12 – due Thursday, February 4, 2021

1. Let K be the field $\mathbb{Q}(\sqrt{-5})$.
   (a) Find a $\mathbb{Z}/4\mathbb{Z}$-extension L/K such that $\sqrt{-1} \in L$. (Hint: take L to be abelian over $\mathbb{Q}$.)
   (b) Check that this does not contradict PS 9, problem 3(a).

2. Put $K = \mathbb{Q}_p(\sqrt{p})$ for some $p > 2$. Compute the Herbrand quotient of K^* as a G-module for $G = \mathrm{Gal}(\mathbb{Q}_p(\sqrt{p})/\mathbb{Q}_p)$. (Hint: use the exact sequence $0 \to \mathcal{O}_K^* \to K^* \to \mathbb{Z} \to 0$.)

3. Let G be a profinite group and let n be a positive integer. Prove that the following two statements are equivalent.
   (a) For every open normal subgroup H of G, the group G/H is generated by some subset of at most n elements.
   (b) There exist elements $g_1, \dots, g_n \in G$ such that for every open normal subgroup H of G, the group G/H is generated by the images of $g_1, \dots, g_n$.

   (Hint: use the fact that by Tykhonov's theorem, an inverse limit of nonempty finite sets is nonempty whether or not the transition maps are surjective.)

4. Let G be the group $\mathrm{GL}_n(\mathbb{Z}_p)$ for some positive integer n.
   (a) Prove that the map $G \to \varprojlim_m \mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})$ is an isomorphism. This means that G can be viewed as a profinite group.
   (b) Describe a Sylow p-subgroup of G in the sense of CFT exercise 3.4.4.4. (Hint: this is basically the same thing as describing a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})$. Try the cases $m = 1$ and $m = 2$ first to get the idea.)

5. In class, it was stated that the local reciprocity map for $K = \mathbb{Q}_p$ is the map

$$\mathbb{Q}_p^* \cong \mathbb{Z}_p^* \times p^{\mathbb{Z}} \to \mathbb{Z}_p^* \times \prod_{q \neq p} \mathbb{Z}_q^* \cong \mathrm{Gal}(K_1/\mathbb{Q}_p) \times \mathrm{Gal}(K_2/\mathbb{Q}_p)$$

   where $K_1 = \bigcup_n \mathbb{Q}_p(\zeta_{p^n})$ and $K_2 = \bigcup_n \mathbb{Q}_p(\zeta_{p^n-1})$. Check that this map satisfies condition (b) of the local reciprocity law (CFT Theorem 4.1.2) for the extension $L = \mathbb{Q}_p(\zeta_p)$. (Optional: also try $L = \mathbb{Q}_p(\zeta_{p^n})$.)