Math 204B (Number Theory), UCSD, winter 2021
Problem Set 13 – due Thursday, February 11, 2021

1. Let K be a finite extension of $\mathbb{Q}_p$.
   (a) Show that K^* is a locally compact abelian group for the subspace topology on $K \times K$ via the map $x \mapsto (x, x^{-1})$.
   (b) Show that K^* is closed in $K \times K$ but not in K. This distinction will show up more seriously in the definition of the idèles.

2. Let L/K be an extension of finite fields. Prove directly (without using any group cohomology) that the norm map $\mathrm{Norm}_{L/K} : L^* \to K^*$ is surjective. (Hint: write an explicit formula for the norm map.)

3. Let G be a finite group. Let M be a G-module whose underlying abelian group is a $\mathbb{Q}$-vector space. Prove that M is acyclic. (Hint: first use the description using cochains to show that the groups $H^i(G, M)$ are themselves divisible. Then combine with the fact that the cohomology groups are killed by $\#G$; see CFT Example 3.2.21.)

4. Let K be a finite extension of $\mathbb{Q}_p$.
   (a) Show that for any positive integer n, $K^*/(K^*)^n$ is a finite group.
   (b) Let L be a finite extension of K. Show that $\mathrm{Norm}_{L/K} L^*$ is an open subgroup of K^* of finite index. (Hint: restrict the norm map to K^*, then apply (a).)

5. Let p be an odd prime. Let R_1 and R_2 be quaternion algebras over $\mathbb{Q}_p$ (see CFT exercise 4.1.5). Show directly (without local class field theory) that if neither R_1 nor R_2 is isomorphic to $M_2(\mathbb{Q}_p)$, then they are isomorphic to each other. (Hint: first recall the structure of $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$.) Optional: extend this to $p = 2$.