Math 204B (Number Theory), UCSD, winter 2021
Problem Set 16 – due Thursday, March 4, 2021

1. For S a finite set of places of a number field K, let $\mathbb{A}_S$ be the set of $x \in \mathbb{A}_K$ which are integral at all finite places of K (i.e., the "ring of adelic S-integers"). Prove that $K + \mathbb{A}_S = \mathbb{A}_K$. (Hint: apply the Chinese remainder theorem.)

2. (a) Prove that the set of $x \in \mathbb{A}_{\mathbb{Q}}$ such that for all $c \in \mathbb{Q}$, cx is a sum of three squares is the kernel of the projection $\mathbb{A}_{\mathbb{Q}} \to \mathbb{R} \times \mathbb{Q}_2$.

   (b) Let p be an odd prime. Choose $a \in \mathbb{Z}_{(p)}^*$ whose reduction modulo p is not a quadratic residue. Prove that the set of $x \in \mathbb{A}_{\mathbb{Q}}$ such that for all $c \in \mathbb{Q}$, cx is in the image of $(y, z, w) \mapsto y^2 - az^2 + pw^2$ is the kernel of the projection $\mathbb{A}_{\mathbb{Q}} \to \prod_{v \in S} \mathbb{Q}_v$ for some finite set S of places of $\mathbb{Q}$ containing v. (Hint: use the fact that if q is an odd prime and n is an integer not divisible by p, then the map $(y, z) \mapsto y^2 + nz^2$ from $\mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ is surjective. There is a clever proof of this using the pigeonhole principle, but you don't need to include this.)

3. (a) Let v, w be distinct places of $\mathbb{Q}$. Prove that the fields $\mathbb{Q}_v$ and $\mathbb{Q}_w$ are not isomorphic even if you ignore their topologies. (Hint: prove that the set of $x \in \mathbb{Q}$ which are squares in $\mathbb{Q}_v$ is not the same as the set of $x \in \mathbb{Q}$ which are squares in $\mathbb{Q}_w$.)

   (b) Let S be a finite set of places of $\mathbb{Q}$. Using (a) and the fact that each completion $\mathbb{Q}_v$ has trivial automorphism group as a bare ring (see PS 7, problem 7), prove that the product $\prod_{v \in S} \mathbb{Q}_v$ has trivial automorphism group as a bare ring.

4. Let K be a number field. Prove that the integral closure of $\mathbb{Q}$ in $\mathbb{A}_K$ equals K. (Hint: otherwise, there exists a number field L properly containing K such that $L \subseteq \mathbb{A}_K$. Now use the fact that there are infinitely many primes of K that do not split completely in L to deduce a contradiction.)

5. Using the previous exercises, prove the following.

   (a) The ring $\mathbb{A}_{\mathbb{Q}}$ has trivial automorphism group (as a bare ring).

   (b) (Optional) For any *totally real* Galois number field K, the automorphism group of $\mathbb{A}_K$ (as a bare ring) is $\mathrm{Gal}(K/\mathbb{Q})$. (See Zulip for hints.)