Math 204B (Number Theory), UCSD, winter 2021
Problem Set 17 – due Thursday, March 11, 2021

If you are planning to submit a final project, these will be due by Friday, March 19.

1. Let L/K be a Galois extension of number fields with Galois group G.
   (a) Give an explicit example for which $\mathrm{Cl}(K) \to \mathrm{Cl}(L)^G$ fails to be injective. (Hint: use the principal ideal theorem.)
   (b) Give an explicit example for which $\mathrm{Cl}(K) \to \mathrm{Cl}(L)^G$ fails to be surjective. (Hint: take L to be a quadratic extension of $\mathbb{Q}$ with $\mathrm{Cl}(L) \cong \mathbb{Z}/4\mathbb{Z}$.)

2. Put $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$. Show that $5^2 \in \mathrm{Norm}_{K_w/\mathbb{Q}_v} K_w^*$ for every place w of K lying above a place v of $\mathbb{Q}$, but $5^2 \notin \mathrm{Norm}_{K/\mathbb{Q}} K^*$.

3. Prove the following special case of the *Hasse-Minkowski theorem*. Let K be a number field and choose $a, b, c \in K^*$. Suppose that for each place v of K, there exist $x, y, z \in K_v$, not all zero, such that $ax^2 + by^2 + cz^2 = 0$. Then there exist $x, y, z \in K$, not all zero, such that $ax^2 + by^2 + cz^2$. (Hint: show that $-c$ is a norm from $K(\sqrt{-b/a})$ to K.)

4. (a) Let K be a field of characteristic not equal to 2. Show that 16 is an 8th power in K if and only if one of $-1, 2, -2$ is a square in K.
   (b) Put $K = \mathbb{Q}(\sqrt{7})$. Show that 16 is an 8th power in every completion of K, but not in K itself.

5. Let K be a number field, let S be a finite set of finite places of K, and let m be a positive integer. Prove that there exists a cyclic extension L of K contained in $K(\zeta_n)$ for some n such that for all $v \in S$, for some place w of L above K, $[L_w : K_v]$ is divisible by m.