Math 204B (Number Theory), UCSD, winter 2021
Problem Set 9 – due Thursday, January 14, 2021

Some reminders: solutions should be submitted via CoCalc (same project as for 204A). Place your solutions in the folder assignments/2021-01-14/ in your course project. Collaboration and research is fine, as long as you do the following.

• Try the problems yourself first!
• Write the solutions in your own words.
• Acknowledge all sources (textbooks, results from class or previous problem sets, etc.) and collaborators.

New for 204B: please request all extensions in advance of the due date. Remember, you only need to submit 6 of 9 complete problem sets for full credit (or one fewer if you choose the final project option).

1. Let \( n \) be a positive integer and let \( K \) be a field of characteristic coprime to \( n \). (Note that we are not assuming that \( \zeta_n \in K \).)

   (a) Show that for any \( a \in K^\times \), \([K(a^{1/n}) : K]\) is a divisor of \( n \). (Hint: use induction to reduce to the case where \( n \) is prime, then compare what happens over \( K \) and \( K(\zeta_n) \).)

   (b) Show that \([K(a^{1/n}) : K] = n\) if and only if \( a \notin (K^\times)^p \) for each prime divisor \( p \) of \( n \). (Hint: there is a reduction to the case where \( n = p^2 \) and \( \zeta_p \in K \). In this case, distinguish according to whether or not \( \zeta_n \in K \).)

2. Prove Dedekind’s lemma on linear independence of automorphisms: if \( L/K \) is a field extension and \( g_1, \ldots, g_n \) are distinct automorphisms of \( L \) over \( K \), then there do not exist \( x_1, \ldots, x_n \in L \) such that \( x_1g_1(y) + \cdots + x_ng_n(y) = 0 \) for all \( y \in L \). (Hint: choose a counterexample with \( n \) minimal, noting that necessarily \( n > 1 \). Since \( g_1 \neq g_2 \), we can find an element \( z \in L^\times \) such that \( g_1(z) \neq g_2(z) \). We can then modify our original relation either by multiplying on the outside by \( z \) or by rescaling the variable \( y \) by \( z \); compare these and obtain a contradiction to the choice of \( n \).)

3. (a) Let \( K \) be a field of characteristic zero not containing \( \sqrt{-1} \). Suppose that there exists a Galois extension \( L \) of \( K \) containing \( \sqrt{-1} \) with Galois group \( \mathbb{Z}/4\mathbb{Z} \). Prove that there must exist \( x, y \in K \) such that \( x^2 + y^2 = -1 \). (Hint: write \( L \) as \( K(\sqrt{-1}, a^{1/2}) \) for some \( a \in K(\sqrt{-1}) \). Let \( \sigma \) be a generator of \( \text{Gal}(L/K) \) and write \( \sigma(a)/a = (x + y\sqrt{-1})^2 \).)

   (b) Show that no such extension can exist when \( K = \mathbb{Q}_2 \).

4. (a) Show that there is no Galois extension of \( \mathbb{Q}_2 \) with Galois group \((\mathbb{Z}/4\mathbb{Z})^3\). (Hint: separate cases based on whether or not this extension contains \( \sqrt{-1} \), and remember the classification of quadratic extensions of \( \mathbb{Q}_2 \) from PS 8.)
(b) Let $K$ be a Galois extension of $\mathbb{Q}_2$ such that $\text{Gal}(K/\mathbb{Q}_2)$ is an abelian group of order a power of 2. Prove that $K$ is contained in $\mathbb{Q}(\zeta_N)$ for some positive integer $N$.

5. Let $p$ be an odd prime. Let $U_1$ be the set of elements of $\mathbb{Z}_p[\zeta_p]$ congruent to 1 modulo $\pi = 1 - \zeta_p$. Prove that $U_1^p$ equals the set of elements of $\mathbb{Z}_p[\zeta_p]$ congruent to 1 modulo $\pi^{p+1}$. (Hint: use the binomial series as in PS 7 problem 7(b).)