**Math 204B (Number Theory), UCSD, winter 2025**
**Problem Set 1**

Solutions should be submitted by email, as a PDF which may represented typed text, handwritten text on a tablet, or a photo of handwritten text on paper. Collaboration and research is fine, as long as you do the following.

- Try the problems yourself first!
- Write the solutions in your own words.
- Acknowledge all sources (textbooks, results from class or previous problem sets, etc.) and collaborators.

There is no fixed deadline for this or any other problem set (except for the end of the quarter). However, the sooner you submit it the sooner you will get feedback; and you *will* be allowed to resubmit problem sets after receiving feedback if you wish.

For this problem set only: please *do not* assume any results from class field theory.

1. Let $L/K$ be an extension of number fields. Prove that $\operatorname{rank} \mathcal{O}_L^\times > \operatorname{rank} \mathcal{O}_K^\times$ *unless* either
   - $L = K$, or
   - $K$ is a totally real field and $L = K(\sqrt{\alpha})$ where $\alpha \in K^\times$ is totally negative.

   (Hint: apply Dirichlet's units theorem.)

2. Let $p$ be an odd prime. Prove that the unit group $\mathbb{Z}[\zeta_p]^\times$ is generated by $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]^\times$ together with $\zeta_p$.

3. Prove that every quadratic extension of $\mathbb{Q}(i)$ is ramified over at least one prime. (Hint: write a general quadratic extension $K$ of $\mathbb{Q}(i)$ as $\mathbb{Q}(i, \sqrt{D})$ where $D \in \mathbb{Z}[i]$ is squarefree. Show that $K$ ramifies over every prime factor of $\sqrt{D}$ with odd norm; then analyze the remaining cases by hand.)

4. Find a quadratic extension of $\mathbb{Q}(\sqrt{-13})$ which is unramified above every prime. (Hint: the case of $\mathbb{Q}(\sqrt{-5})$ is explained in Example 2.1.1 of my CFT notes.)

5. Let $P(x) = x^5 - x + 1$. Using the fact that this polynomial has discriminant $2869 = 19 \times 151$, show that its splitting field is an everywhere unramified extension of $\mathbb{Q}(\sqrt{2869})$ with Galois group $A_5$. (This kind of example *will not* be explained by class field theory.)