Math 204B (Number Theory), UCSD, winter 2025
Problem Set 2

From now on, you may use freely the statements of class field theory unless otherwise specified.

1. Identify the Galois group of $\mathbb{Q}(\zeta_{25})/\mathbb{Q}$ with $(\mathbb{Z}/25\mathbb{Z})^\times$ via the Artin map, and let K be the subfield of $\mathbb{Q}(\zeta_{25})$ fixed by the image of 7 in $(\mathbb{Z}/25\mathbb{Z})^\times$. Show that p splits completely in K if and only if $p \equiv \pm 1, \pm 7 \pmod{25}$.

2. Let K be a quadratic number field. Show *without* using class field theory that K has an abelian extension which is not contained in $K(\zeta_n)$ for any positive integer n. (Hint: show that there exists $u \in K^\times$ such that $K(\sqrt{u})$ is not Galois over $\mathbb{Q}$.)

3. Let K be a totally complex number field (i.e., all of its archimedean places are complex) which is a quadratic extension of a totally real number field K^+. Prove that the class number of h_K is divisible by the class number of h_{K^+}. (Hint: see CFT notes 2.1, exercise 3.)

4. Find the ray class field of $\mathbb{Q}(\sqrt{-3})$ of conductor (5), and verify Artin reciprocity explicitly in this case. (Hint: first check that the ray class group is cyclic of order 4. You may find it useful to search through LMFDB to come up with a guess for the correct extension, but you should then check that the result is correct.)

5. Let G be a finite group and let H be its commutator subgroup. Prove that the transfer map $V\colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is zero. (Hint: see CFT notes 2.3, exercise 7. You may assume without proof the results of CFT notes 2.3, exercises 5 and 6.)