Math 203C (Number Theory), UCSD, spring 2015
Artin L-functions

Let K be a number field. Let G be a group equipped with a linear representation $\rho : G \to \mathrm{GL}(V)$ on some finite-dimensional $\mathbb{C}$-vector space V. Suppose that for each maximal ideal $\mathfrak{p}$ of $\mathfrak{o}_K$ (possibly with finitely many exceptions), we have been given an element $g_\mathfrak{p}$ of G. We can then form an associated L-function by taking the product

$$L(s, \rho, \{g_\mathfrak{p}\}) = \prod_\mathfrak{p} \det(1 - \mathrm{Norm}(\mathfrak{p})^{-s} g_\mathfrak{p}, V)^{-1};$$

this is absolutely convergent for $\mathrm{Re}(s) > 1$, and only depends on the conjugacy classes of the elements $g_\mathfrak{p}$. Note that a direct sum of representations corresponds to a product of L-functions.

For example, let χ be a Dirichlet character. Then the Dirichlet L-function associated to χ arises in this fashion by taking $G = \mathbb{Z}/n\mathbb{Z}$ and $V = \mathbb{C}$.

For another example, take $K = \mathbb{Q}$, let L/K be a number field, let G be the Galois group of the Galois closure of L/K, and let ρ be the natural permutation representation associated to L/K. (If L/K is Galois, this is the permutation representation. Otherwise, it's the action on left cosets of the subgroup of G fixing L.) Then one gets the Dedekind zeta function associated to K (up to finitely many Euler factors).

More generally, let L/K be a finite Galois extension, take $G = \mathrm{Gal}(L/K)$, and let ρ be any linear representation of G. For each $\mathfrak{p}$ which is unramified over K, choose a prime $\mathfrak{q}$ above $\mathfrak{p}$, and let $g_\mathfrak{p}$ be Artin's *Frobenius element* of G associated to $\mathfrak{q}$, namely the element of the decomposition group of $\mathfrak{q}$ corresponding to the $\mathrm{Norm}(\mathfrak{p})$-power Frobenius map on $\mathfrak{o}_L/\mathfrak{q}$. Then the product one gets is the *Artin L-function.* (One should add some Euler factors corresponding to ramified primes; these work the same way, except that Frobenius elements are only well-defined modulo the inertia group of $\mathfrak{q}$, so one replaces V with the subspace fixed by inertia.)

Theorem 1. *Any Artin L-function has meromorphic continuation to $\mathbb{C}$, and satisfies a functional equation relating its values at s and $1 - s$ (involving suitable Euler factors at infinite places).*

One can say something about the order at $s = 1$: it is equal to the dimension of the fixed space V^G, i.e., the multiplicity of the trivial representation. If this is zero, one expects to get analytic continuation to $\mathbb{C}$, but this is only known in a few cases, notably when L/K is solvable (Langlands solvable base change theorem). The meromorphic continuation is derived from such special cases using Brauer's theorem on induced characters.

Suppose that for a given L/K, one had meromorphic continuation of all of the Artin L-functions to a neighborhood of $\mathrm{Re}(s) \geq 1$ *and* no zeroes or poles on $\mathrm{Re}(s) = 1$ except the ones at $s = 1$. Then one could imitate the proof of the prime number theorem to prove the *Chebotarev density theorem*: the elements $g_\mathfrak{p}$ are uniformly distributed in G. However, this can be proved unconditionally once one has class field theory and the Langlands theorem (or really just the case of abelian representations).

To follow: another example of this framework, for elliptic curves (the Sato-Tate conjecture).