Math 203C (Number Theory), UCSD, spring 2015

Artin L-functions

Let \( K \) be a number field. Let \( G \) be a group equipped with a linear representation \( \rho : G \to \text{GL}(V) \) on some finite-dimensional \( \mathbb{C} \)-vector space \( V \). Suppose that for each maximal ideal \( \mathfrak{p} \) of \( \mathcal{O}_K \) (possibly with finitely many exceptions), we have been given an element \( g_\mathfrak{p} \) of \( G \). We can then form an associated \( L \)-function by taking the product

\[
L(s, \rho, \{g_\mathfrak{p}\}) = \prod_\mathfrak{p} \det(1 - \text{Norm}(\mathfrak{p})^{-s}g_\mathfrak{p}, V)^{-1};
\]

this is absolutely convergent for \( \text{Re}(s) > 1 \), and only depends on the conjugacy classes of the elements \( g_\mathfrak{p} \). Note that a direct sum of representations corresponds to a product of \( L \)-functions.

For example, let \( \chi \) be a Dirichlet character. Then the Dirichlet \( L \)-function associated to \( \chi \) arises in this fashion by taking \( G = \mathbb{Z}/n\mathbb{Z} \) and \( V = \mathbb{C} \).

For another example, take \( K = \mathbb{Q} \), let \( L/K \) be a number field, let \( G \) be the Galois group of the Galois closure of \( L/K \), and let \( p \) be the natural permutation representation associated to \( L/K \). (If \( L/K \) is Galois, this is the permutation representation. Otherwise, it’s the action on left cosets of the subgroup of \( G \) fixing \( L \).) Then one gets the Dedekind zeta function associated to \( K \) (up to finitely many Euler factors).

More generally, let \( L/K \) be a finite Galois extension, take \( G = \text{Gal}(L/K) \), and let \( \rho \) be any linear representation of \( G \). For each \( \mathfrak{p} \) which is unramified over \( K \), choose a prime \( q \) above \( \mathfrak{p} \), and let \( g_\mathfrak{p} \) be Artin’s Frobenius element of \( G \) associated to \( q \), namely the element of the decomposition group of \( q \) corresponding to the \( \text{Norm}(\mathfrak{p}) \)-power Frobenius map on \( \mathcal{O}_L/q \). Then the product one gets is the Artin \( L \)-function. (One should add some Euler factors corresponding to ramified primes; these work the same way, except that Frobenius elements are only well-defined modulo the inertia group of \( q \), so one replaces \( V \) with the subspace fixed by inertia.)

**Theorem 1.** Any Artin \( L \)-function has meromorphic continuation to \( \mathbb{C} \), and satisfies a functional equation relating its values at \( s \) and \( 1 - s \) (involving suitable Euler factors at infinite places).

One can say something about the order at \( s = 1 \): it is equal to the dimension of the fixed space \( V^G \), i.e., the multiplicity of the trivial representation. If this is zero, one expects to get analytic continuation to \( \mathbb{C} \), but this is only known in a few cases, notably when \( L/K \) is solvable (Langlands solvable base change theorem). The meromorphic continuation is derived from such special cases using Brauer’s theorem on induced characters.

Suppose that for a given \( L/K \), one had meromorphic continuation of all of the Artin \( L \)-functions to a neighborhood of \( \text{Re}(s) \geq 1 \) and no zeroes or poles on \( \text{Re}(s) = 1 \) except the ones at \( s = 1 \). Then one could imitate the proof of the prime number theorem to prove the Chebotarev density theorem: the elements \( g_\mathfrak{p} \) are uniformly distributed in \( G \). However, this can be proved unconditionally once one has class field theory and the Langlands theorem (or really just the case of abelian representations).
To follow: another example of this framework, for elliptic curves (the Sato-Tate conjecture).