Math 203C (Number Theory), UCSD, spring 2015
Dedekind zeta functions

Let K be a number field. We define the *Dedekind zeta function* of K as the Dirichlet series

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}$$

where $a_K(n)$ counts the number of ideals of $\mathfrak{o}_K$ of absolute norm n. We can also write this as a sum

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{Norm}(\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs over nonzero ideals of $\mathfrak{o}_K$. Either way, we get an Euler product factorization

$$\zeta_K(s) = \prod_p \left(\sum_{n=0}^{\infty} \frac{a_K(p^n)}{p^{ns}} \right) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathrm{Norm}(\mathfrak{p})^s} \right)^{-1}$$

where p runs over rational primes and $\mathfrak{p}$ over maximal ideals of $\mathfrak{o}_K$. From any of these expansions, we may see that $\zeta_K(s)$ is defined by an absolutely convergent Dirichlet series for $\mathrm{Re}(s) > 1$.

For example, let $K = \mathbb{Q}(\zeta_m)$ be the m-th cyclotomic field. For each prime p not dividing m, let $\mathfrak{p}$ be a prime of $\mathfrak{o}_K = \mathbb{Z}[\zeta_m]$ lying above p. Then the absolute Frobenius automorphism (p-powering) on $\mathfrak{o}_K/\mathfrak{p}$ takes ζ_m to ζ_m^p. Its order, which is also the degree of $\mathfrak{o}_K/\mathfrak{p}$ over $\mathbb{F}_p$, equals the order of p in the group $(\mathbb{Z}/m\mathbb{Z})^*$. Using this calculation, we may check that

$$\prod_{\mathfrak{p}|p} \left(1 - \frac{1}{\mathrm{Norm}(\mathfrak{p})^s} \right)^{-1} = \prod_{\chi} \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}$$

where χ runs over the Dirichlet characters of level m. In other words, $\zeta_K(s)$ equals the product of the L-functions $L(\chi, s)$ for all Dirichlet characters of level m together with finitely many additional Euler factors. This statement can be interpreted in terms of the representation theory of the group $\mathrm{Gal}(K/\mathbb{Q})$; it naturally generalizes to nonabelian Galois extensions, whose zeta functions factor into *Artin L-functions*. More on these later.

Returning to the general case, one has the following result, which we won't prove right now. (It is best proved using a version of the Poisson summation formula on the group of adèles of K, as in Tate's thesis. See the last chapter of Cassels-Fröhlich.)

Theorem 1. *The function $\zeta_K(s)$ extends to a meromorphic function on all of $\mathbb{C}$ with a simple pole at $s = 1$ and no other poles. It also admits the functional equation $\Phi_K(s) = \Phi_K(1-s)$ where*

$$\Phi_K(s) = \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} 4^{-r_2} |d_K|^{s/2} \pi^{-[K:\mathbb{Q}]s/2} \zeta_K(s)$$

with (as usual) r_1, r_2 being the number of real and complex places of K and d_K being the absolute discriminant of K.

As for the Riemann zeta, one can prove that $\zeta_K(s)$ has no zeroes on the line $\mathrm{Re}(s) = 1$. I leave this as an exercise. The analogue of the Riemann Hypothesis is the following.

Conjecture 2. *The zeroes of* $\zeta_K(s)$ *in the range* $\mathrm{Re}(s) \in (0, 1)$ *all lie on the line* $\mathrm{Re}(s) = 1/2$. *(Again, the only other zeroes are some trivial zeroes at negative integers imposed by the functional equation.)*

I will at least try to say something about the following.

Theorem 3. *The residue of* $\zeta_K(s)$ *at* $s = 1$ *equals*

$$\frac{2^{r_1+r_2}\pi^{r_2}R_K h_K}{\omega_K |d_K|^{1/2}}$$

where R_K *equals the unit regulator of* K, h_K *equals the class number, and* ω_K *equals the number of roots of unity in* K.