# Math 203C (Number Theory), UCSD, spring 2015
## Zeta functions for function fields

Fix a prime number $p$. Let $K$ be a finite extension of the rational function field $\mathbb{F}_p(t)$. As in the number field case, the integral closure $\mathfrak{o}_K$ of $\mathbb{F}_p[t]$ in $K$ is a Dedekind domain, and we may define the *Dedekind zeta function* of $K$ as the Dirichlet series

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}$$

where $a_K(n)$ counts the number of ideals of $\mathfrak{o}_K$ of absolute norm $n$. We can also write this as a sum

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{Norm}(\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs over nonzero ideals of $\mathfrak{o}_K$. We get an Euler product factorization

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathrm{Norm}(\mathfrak{p})^s}\right)^{-1}$$

where $p$ runs over rational primes and $\mathfrak{p}$ over maximal ideals of $\mathfrak{o}_K$. From any of these expansions, we may see that $\zeta_K(s)$ is defined by an absolutely convergent Dirichlet series for $\mathrm{Re}(s) > 1$.

However, in this case one has a more geometric interpretation of this construction. Let $\mathbb{F}_q$ be the integral closure of $\mathbb{F}_p$ in $K$. Then $K$ can be identified with the field of rational functions on a certain smooth affine algebraic curve $C_0$ over $\mathbb{F}_q$. Each prime ideal $\mathfrak{p}$ of $\mathfrak{o}_K$ has norm $q^m$ for some positive integer $m$, so we can rewrite the Dirichlet series for $\zeta_K(s)$ as a power series in $q^{-s}$.

**Lemma 1.** *We have an identity of formal power series in $q^{-s}$:*

$$\zeta_K(s) = \exp\left(\sum_{n=1}^{\infty} \frac{\#C_0(\mathbb{F}_{q^n})}{n} q^{-ns}\right).$$

*Proof.* Each prime ideal $\mathfrak{p}$ of norm $q^m$ gives rise to $m$ distinct points on $\#C_0$ over $\mathbb{F}_{q^m}$, and hence also over $\mathbb{F}_{q^{mn}}$ for every positive integer $n$. Now note that

$$\begin{aligned}
\sum_{\mathfrak{p}} \log\left(1 - \frac{1}{\mathrm{Norm}(\mathfrak{p})^s}\right)^{-1} &= \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Norm}(\mathfrak{p})^{-ns} \\
&= \sum_{m=1}^{\infty} \sum_{\mathfrak{p}:\mathrm{Norm}(\mathfrak{p})=q^m} \sum_{n=1}^{\infty} \frac{m}{mn} q^{-mns} \\
&= \sum_{d=1}^{\infty} \left(\sum_{m|d} \sum_{\mathfrak{p}:\mathrm{Norm}(\mathfrak{p})=q^m} m\right) \frac{1}{n} q^{-ds}.
\end{aligned}$$

$\square$

For example, if $K = \mathbb{F}_q(t)$, then $C_0$ is the affine line over $\mathbb{F}_q$, so $\#C_0(\mathbb{F}_{q^n}) = q^n$ for all $n$. We thus have

$$\zeta_K(s) = \exp\left(\sum_{n=1}^{\infty} \frac{q^n}{n} q^{-ns}\right) = (1 - q^{1-s})^{-1}.$$

In particular, $\zeta_K(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with no zeroes whatsoever! One discrepancy with the Riemann zeta function: there are also poles at $s = 1 + \frac{2\pi i n}{\log q}$ for all $n \in \mathbb{Z}$.

As with Dedekind zeta functions, one can get something with a good functional equation by adding Euler factors corresponding to completions of $K$ restricting to the infinite place of $\mathbb{F}_p(t)$ to get a new function $\xi_K(s)$. The latter is the $\infty$-adic absolute value: $|f|_\infty = p^{\mathrm{ord}_\infty(f)}$. Unlike in the number field case, though, these missing Euler factors have a similar shape as the finite ones; you just add one factor of $(1 - q^{-ms})$ for each infinite place with residue field $\mathbb{F}_{q^m}$. One then has an analogue of Lemma 1 where one counts points on the smooth projective completion $C$ of $C_0$. (For this reason, $\xi_K$ is commonly denoted $\zeta_C$ and is itself called the *zeta function* of the curve $C$. For $K = \mathbb{F}_q(t)$, we get

$$\xi_K(s) = (1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$$

which satisfies $\xi_K(s) = q^{-1}\xi_K(1 - s)$.

**Theorem 1** (Weil)**.** *For any $K$, $\xi_K(s)$ extends to a meromorphic function on $\mathbb{C}$ with simple poles at $s = \frac{2\pi i n}{\log q}$, $s = 1 + \frac{2\pi i n}{\log q}$ and no other poles. There is also a functional equation of the form*

$$\xi_K(1 - s) = q^{a+bs}\xi_K(s)$$

*for certain constants $a, b$.*

Better yet, the analogue of the Riemann hypothesis is a theorem! More on this later.

**Theorem 2** (Weil)**.** *The zeroes of $\xi_K(s)$ all lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

For example, if $C$ is an elliptic curve, then by results of Hasse we have

$$\zeta_K(s) = \frac{(1 - \alpha q^{-s})(1 - \beta q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

for some $\alpha, \beta \in \mathbb{C}$ which are complex conjugates of each other and have product $q$. In general,

$$\zeta_K(s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

where $P(T) = P_0 + P_1 T + \cdots + P_{2g}T^{2g}$ is a polynomial with integer coefficients of degree $2g$, where $g$ is the *genus* of the curve, $P_0 = 1$, $P_{g+i} = q^i P_{g-i}$, and the complex roots of $P$ all have absolute value $q^{1/2}$.