Math 203C (Number Theory), UCSD, spring 2015
The Riemann zeta function

For a complex number s with $\mathrm{Re}(s) > 1$, the quantity $\zeta(s)$ is defined as the complex number computed by the absolutely convergent infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

As noted by Euler, using the unique factorization of positive integers into primes, this expression can be rewritten as an absolutely convergent infinite product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

More facts:

- The *analytic continuation property*: there is a unique extension of ζ to a meromorphic function $\zeta : \mathbb{C} - \{1\} \to \mathbb{C}$ with a simple pole at $s = 1$. The function ζ is called the *Riemann zeta function.*
- The *functional equation property*: the values of ζ at s and $1 - s$ determine each other. The cleanest way to write this is to define a modified zeta function

  $$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

  which then satisfies

  $$\xi(s) = \xi(1 - s).$$

  The extra factors should be thought of as a missing term in Euler's infinite product corresponding to the archimedean place of $\mathbb{R}$.
- *Special values*: it is well known that $\zeta(2) = \pi^2/6$. More generally, for n a positive integer

  $$\zeta(2n) = \frac{(-1)^{n-1}2^{2n-1}B_{2n}}{(2n)!}\pi^{2n}$$

  where B_{2n} are the sequence of *Bernoulli numbers*

  $$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

  This statement has a deeper interpretation in terms of *algebraic K-theory.* For other types of zeta functions, such interpretations will lead to many theorems and conjectures such as *class number formulas*, the *conjecture of Birch and Swinnerton-Dyer*, etc.

- *Prime number theorem*: the aforementioned properties of ζ can be used to prove the usual estimate on the distribution of prime numbers, as originally conjectured by Gauss: the number of prime numbers in the interval $[1, N]$ is asymptotic to $N/\log N$ as $N \to \infty$. (A better approximation than $N/\log N$ is $\int_2^N dx/\log x$.)

- The *Riemann hypothesis*: notice that on one hand, the infinite product for $\zeta(s)$ cannot equal zero for $\mathrm{Re}(s) > 1$; on the other hand, for $\mathrm{Re}(s) < 0$, the only zeroes are the *trivial zeroes* at negative even integers coming from the functional equation (recall that Γ has poles at all negative integers). It is conjectured that all remaining zeroes satisfy $\mathrm{Re}(s) = 1/2$. It is not hard to rule out zeroes with $\mathrm{Re}(s) = 0, 1$, but otherwise it is hard to prove much. This conjecture is equivalent to an improved error term in the prime number theorem, but more on this later.