Math 206A (Topics in Algebraic Geometry): Weil cohomology in practice
Kiran S. Kedlaya, fall 2019
Problem set 1

Recommended reading: Weil, "Number of solutions of equations in finite fields". (Links for recommended readings can be found on the course web site.)

(1) Let k be a finite field of order q and fix an additive character (homomorphism) $\psi : k \to \mathbb{C}^\times$. For $\chi : k^\times \to \mathbb{C}^\times$ a nontrivial multiplicative character, define the Gauss sum

$$G_\psi(\chi) = \sum_{x \in k^\times} \chi(x)\psi(x).$$

Prove that $G_\psi(\chi)G_\psi(\overline{\chi}) = q$, where $\overline{\chi}$ is the character for which $\overline{\chi}(x)$ is the complex conjugate of $\chi(x)$. (Hint: write the product as a sum over $x, y \in k^\times$, then regroup terms by the value of x/y.)

(2) Fix a choice of χ as above. For $P(T) = T^n + P_{n-1}T^{n-1} + \cdots + P_0 \in k[T]$ a monic polynomial, define

$$\lambda(P) = \chi(P_0)\psi(P_{n-1}).$$

(In particular, $\lambda(1) = 1$.) Show that

$$\lambda(P_1P_2) = \lambda(P_1)\lambda(P_2) \qquad (P_1, P_2 \in k[T])$$

and deduce that for each positive integer n, in $\mathbb{C}[\![U]\!]$ we have

$$\sum_{P \in k[T] \text{ monic}} \lambda(P)U^{\deg(P)} = \prod_{Q \in k[T] \text{ monic irreducible}} (1 - \lambda(Q)U^{\deg Q})^{-1}.$$

(3) Show that for n a nonnegative integer,

$$\sum_{P \in k[T] \text{ monic}, \deg(P) = n} \lambda(P)U^{\deg(P)} = \begin{cases} 1 & n = 0 \\ G_\psi(\chi)U & n = 1 \\ 0 & n > 1. \end{cases}$$

(4) With notation as in the previous problem, let k' be an extension of k of degree v. Let $\psi' : k' \to \mathbb{C}^\times$ be the additive character given by $\psi \circ \mathrm{Trace}_{k'/k}$. Given χ, let χ' be the multiplicative character given by $\chi \circ \mathrm{Norm}_{k'/k}$. For $P' \in k'[T]$ monic, define λ' by analogy with λ.

For $P \in k[T]$ monic irreducible, let P' run over the irreducible factors of P in $k'[T]$. Prove that

$$\prod_{P'} (1 - \lambda'(P')U^{v \deg(P')}) = \prod_{\rho=0}^{v-1} (1 - \lambda(P)(e^{2\pi i \rho/v}U)^{\deg(P)}).$$

(Hint: let $-\xi$ be a root of one of the factors P', and consider the field extensions $k(\xi)/k$ and $k'(\xi)/k'$.)

(5) Using all of the above, deduce the Davenport-Hasse relation

$$-G_{\psi'}(\chi') = (-G_\psi(\chi))^v.$$