Math 206A (Topics in Algebraic Geometry): Weil cohomology in practice
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Problem set 2

Throughout, let $\mathbb{F}_q$ denote a finite field of characteristic p.

(1) For X an algebraic variety over $\mathbb{F}_q$, we write the zeta function of X as $Z(X, q^{-s})$ for

$$Z(X,T) = \prod_{x \in X^\circ} (1 - T^{\deg(x)})^{-1},$$

where X° denotes the set of Galois orbits of $\overline{\mathbb{F}_q}$-points and $\deg(x)$ is the cardinality of such an orbit. Prove that in $\mathbb{Q}[\![T]\!]$, we have the equality

$$Z(X,T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{q^n})\right).$$

(2) For X equal to the n-dimensional projective space over $\mathbb{F}_q$, compute that

$$Z(X,T) = \frac{1}{(1-T)(1-qT)\cdots(1-q^nT)}.$$

(3) Prove that the following statements are equivalent.
  (i) The power series $Z(X,T)$ represents a rational function in T.
  (ii) There exist $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in \mathbb{C}$ such that

$$\#X(\mathbb{F}_{q^n}) = \alpha_1^n + \cdots + \alpha_r^n - \beta_1^n - \cdots - \beta_s^n \qquad (n = 1, 2, \dots).$$

(4) Let X be the *Grassmannian* of k-dimensional subspaces of m-space over $\mathbb{F}_q$.
  (i) Compute $\#X(\mathbb{F}_{q^n})$; your answer should be a polynomial in q^n depending on k and m. (Hint: count bases of subspaces, then divide by the number of bases of a given subspace.)
  (ii) Compute $Z(X,T)$.

(5) Choose $a_0, \dots, a_r \in \mathbb{F}_q^\times$. For d a positive integer dividing $q-1$, let X_d be the projective hypersurface $a_0x_0^d + \cdots + a_rx_r^d = 0$.
  (i) Let G_d be the group of homomorphisms $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ of order d. For $\chi \in G_d$, extend the definition of χ to $\mathbb{F}_q$ by setting $\chi(0) = 1$ if $\chi = 1$ and $\chi(0) = 0$ otherwise. Show that

$$1 + (q-1)\#X_d(\mathbb{F}_q) = \sum_{(u_0,\dots,u_r)\in X_1} \sum_{\chi_0,\dots,\chi_r \in G_d} \prod_{i=0}^{r} \chi_i(u_i).$$

  (ii) Show that $\chi_0, \dots, \chi_r \in G_d$ are neither all equal to 1 or all distinct from 1, then

$$\sum_{(u_0,\dots,u_d)\in X_1} \prod_{i=0}^{r} \chi_i(u_i) = 0.$$

  (iii) Let T be the set of tuples $(\chi_0, \dots, \chi_r) \in G_d \setminus \{1\}$ with $\chi_0 \cdots \chi_r = 1$. For $(\chi_0, \dots, \chi_r) \in T$, define the *Jacobi sum*

$$j(\chi_0, \dots \chi_r) = \frac{1}{q-1} \sum_{u_0,\dots,u_r \in \mathbb{F}_q : u_0 + \cdots + u_r = 0} \chi_0(u_0) \cdots \chi_r(u_r).$$

Deduce from above that

$$\#X_d(\mathbb{F}_q) = 1 + q + \cdots + q^{r-1} + \sum_{(\chi_0,\cdots,\chi_r)\in T} \chi_0(a_0^{-1}) \cdots \chi_r(a_r^{-1}) j(\chi_0, \dots, \chi_r).$$

(iv) Fix an additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$. Show that

$$j(\chi_0, \dots, \chi_r) = \frac{1}{q} G(\chi_0, \psi) \cdots G(\chi_r, \psi)$$

where $G(\chi, \psi)$ denotes the Gauss sum.

(6) Keep notation as in the previous exercise, but assume only that d is not divisible by p (not that it divides $q-1$).

(i) Show that $\#X_d(\mathbb{F}_q) = \#X_e(\mathbb{F}_q)$ for $e = \gcd(d, q-1)$.

(ii) Using the Davenport-Hasse relation, show that the rationality, functional equation, and Riemann hypothesis hold for $Z(X_d, T)$.