Math 206A (Topics in Algebraic Geometry): Weil cohomology in practice
Kiran S. Kedlaya, fall 2019
Problem set 3

Throughout, let $\mathbb{F}_q$ denote a finite field of characteristic p. You may assume the Weil conjectures for curves and abelian varieties unless otherwise specified

(1) Let X be a nonzero abelian variety over $\mathbb{F}_q$. Prove that if $q \geq 5$, the group $X(\mathbb{F}_q)$ is nontrivial.

(2) Let X be a curve over $\mathbb{F}_q$ such that $\#X(\mathbb{F}_q) = 1$.

  (a) If $q = 3$ or $q = 4$, prove that
$$Z(X,T) = \frac{1 - qT + qT^2}{(1-T)(1-qT)}.$$

  (b) If $q = 2$, prove that the genus of X is at most 4, and that there are at most 6 possibilities for $Z(X,T)$.

  (c) Optional: show that each of the 8 possibilities occurs for a unique X up to isomorphism.

(3) Let X be an abelian variety of dimension g over $\mathbb{F}_q$. Assuming only the existence of complex numbers $\alpha_1, \dots, \alpha_{2g}$ such that
$$X(\mathbb{F}_{q^n}) = (1 - \alpha_1^n) \cdots (1 - \alpha_{2g}^n) \qquad (n = 1, 2, \dots),$$
compute $Z(X,T)$.

(4) Using the Honda-Tate theorem, prove that if A_1, A_2 are abelian varieties over $\mathbb{F}_q$ and $P_1(A_1,T)$ divides $P_1(A_2,T)$, then A_1 is isogenous to the product of A_2 with some other abelian variety.

(5) Let X be a curve of genus g over $\mathbb{F}_q$. Prove the following refinement of the Weil bound due to Serre:
$$|\#X(\mathbb{F}_q) - q - 1| \leq g\lfloor 2\sqrt{q} \rfloor.$$
Hint: apply AM-GM to the numbers $\lfloor 2\sqrt{q}\rfloor + 1 + \alpha + \overline{\alpha}$ where α runs over the Frobenius eigenvalues.