**Math 206A (Topics in Algebraic Geometry): Weil cohomology in practice**
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**Problem set 4**

(1) Let $K$ be a number field. Using the Chebotarev density theorem, prove that the Frobenius elements corresponding to maximal ideals of $\mathfrak{o}_K$ are dense in the absolute Galois group $G_K$. (This is just an exercise in unwinding the definitions.)

(2) In this exercise, we prove the theorem of Borel stated in class on November 3.

(a) Let $f(T) = \sum_{n=0}^{\infty} a_n T^n$ be a power series over an arbitrary field $K$. Prove that $f(T)$ represents a rational function over $K$ if and only if for some positive integer $m$, the determinants of the $(m+1) \times (m+1)$ matrices $A_{n,m} = (a_{n+i+j})_{i,j=0}^m$ vanish for all sufficiently large $n$.

(b) Let $f(T) = \sum_{n=0}^{\infty} a_n T^n$ be a power series over $\mathbb{Z}$. Let $r > 0$ be a real number such that over $\mathbb{Q}_p$, there exists a polynomial $P(T)$ of degree $d < m$ such that $P(T)f(T)$ converges for $|T| < r + \epsilon$ for some $\epsilon > 0$. (We do not assume that $P$ has coefficients in $\mathbb{Z}$.) Prove that for some $C > 0$, $|\det(A_{n,m})|_p \leq Cr^{-n(m-d)}$ for all $n$.

(c) Let $f(T) = \sum_{n=0}^{\infty} a_n T^n$ be a power series over $\mathbb{Z}$. Let $R$ and $r$ be real numbers with $Rr > 1$ such that over $\mathbb{C}$, $f(T)$ converges for $|T| < R$; and over $\mathbb{Q}_p$, $f(T)$ is the ratio of two series that converge for $|T| < r$. Prove that $f$ represents a rational function. (Hint: apply (b) with $r$ replaced by $r - \epsilon$ for which $(R-\epsilon)(r-\epsilon) > 1$, then combine with a trivial bound on $|\det(A_{n,m})|_\infty$.)

(3) Let $\pi$ be an element of an algebraic closure of $\mathbb{Q}_p$ satisfying $\pi^{p-1} = -p$. (You may use without proof the fact that $\mathbb{Z}_p[\pi]$ is a discrete valuation ring with maximal ideal $(\pi)$.) Define the power series

$$E_\pi(T) = \exp(\pi(T - T^p)) \in \mathbb{Q}_p(\pi)[\![T]\!].$$

(a) Prove that $E_\pi(T) \in 1 + \pi\mathbb{Z}_p[\pi][\![T]\!]$.

(b) Prove that $E_\pi(T)$ has radius of convergence strictly greater than 1. In particular, it makes sense to evaluate it at any element of $\mathbb{Z}_p[\pi]$.

(c) Prove that if $t \in \mathbb{Z}_p$ satisfies $t^p = t$, then $E_\pi(t)^p = 1$. (Hint: check that in the identity

$$E_\pi(T)^p = \exp(\pi p T)\exp(-\pi p T^p)$$

it is valid to substitute $t$ *separately* into the two factors on the right.)

(4) With notation as in the previous problem, let $n$ be a positive integer and define

$$E_n(T) := \exp(\pi(T - T^{p^n})) = E_\pi(T)E_\pi(T^p)\cdots E_\pi(T^{p^{n-1}}) \in \mathbb{Q}_p(\pi)[\![T]\!].$$

Show that the formula $t \mapsto E_n([t])$ defines a nontrivial additive character on $\mathbb{F}_{p^n}$, where $[t]$ denotes the unique element of $\mathbb{Z}_{p^n}$ (the finite étale extension of $\mathbb{Z}_p$ with residue field $\mathbb{F}_{p^n}$) lifting $t$ and satisfying $t^{p^n} = t$.

(5) Set $q = p^n$ and let

$$f = \sum_{I=(i_1,\dots,i_d)} a_I x_1^{i_1} \cdots x_d^{i_d} \in \mathbb{F}_q[x_1, \dots, x_d]$$

be a polynomial. Prove that for any positive integer $m$, the number of points $(x_1, \ldots, x_d) \in (\mathbb{F}_{q^m}^\times)^d$ for which $f(x_1, \ldots, x_d) = 0$ equals

$$\frac{(q^m-1)^d}{q^m}\left(1 + (q^m-1)\sum_{x_0,\ldots,x_d \in \mathbb{F}_{q^m}^\times} \prod_{I: a_I \neq 0} \prod_{j=0}^{m-1} E_\pi(a_I([x_0][x_1]^{i_1}\cdots[x_d]^{i_d})^{q^j})\right).$$