Math 206A (Topics in Algebraic Geometry): Weil cohomology in practice
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Problem set 5

(1) Define the rings

$$R = \mathbb{Z}[x_1, y_1, x_2, y_2, \dots], \quad R' = \mathbb{Q}[x_1, y_1, x_2, y_2, \dots], \quad F = \mathrm{Frac}(R) = \mathrm{Frac}(R').$$

Define the power series $x = 1 + x_1 T + x_2 T^2 + \cdots$, $y = 1 + y_1 T + y_2 T^2 + \cdots$, and

$$f = 1/\exp(\log(1/x) \star \log(1/y)) \in R'[\![T]\!]$$

where $\star$ denotes the *Hadamard product*:

$$(a_1 T + a_2 T^2 + \cdots) \star (b_1 T + b_2 T^2 + \cdots) = a_1 b_1 T + a_2 b_2 T^2 + \cdots$$

(a) Let V_1, V_2 be two finite-dimensional vector spaces over F equipped with endomorphisms φ_1, φ_2 satisfying, for some positive integer n,

$$\det(1 - \varphi_1 T, V_1)^{-1} \equiv 1 + x_1 T + \cdots + x_n T^n \pmod{T^{n+1} F[\![T]\!]},$$
$$\det(1 - \varphi_2 T, V_2)^{-1} \equiv 1 + y_1 T + \cdots + y_n T^n \pmod{T^{n+1} F[\![T]\!]},$$

Prove that

$$\det(1 - (\varphi_1 \otimes \varphi_2) T, V_1 \otimes_F V_2)^{-1} \equiv f \pmod{T^{n+1} F[\![T]\!]}.$$

(Hint: pass to an algebraic closure of F and write everything in terms of eigenvalues. Remember that f is determined mod $T^{n+1} F[\![T]\!]$ by $x_1, \dots, x_n, y_1, \dots, y_n$.)

(b) Deduce that $f \in R[\![T]\!]$.

(2) Using the previous exercise, prove that there is a unique functor Λ from rings to rings with the following properties.

(a) The underlying functor from rings to additive groups takes R to $\Lambda(R) = 1 + TR[\![T]\!]$ with the usual series multiplication.

(b) For any ring R, the multiplication map $*$ on $\Lambda(R)$ satisfies

$$(1 - aT)^{-1} * (1 - bT)^{-1} = (1 - abT)^{-1} \qquad (a, b \in R).$$

The ring $\Lambda(R)$ is (a form of) the ring of *big Witt vectors* with coefficients in R.

(3) Let X_1, X_2 be two varieties over $\mathbb{F}_q$. Prove that in $\Lambda(\mathbb{Z})$, we have

$$Z(X_1 \times_{\mathbb{F}_q} X_2, T) = Z(X_1, T) * Z(X_2, T).$$

(4) Let K be a field of characteristic 0. Let $P(x) \in K[x]$ be a monic polynomial of degree $2g + 1$ with no repeated roots.

(a) Let X be the affine scheme $\mathrm{Spec}\, K[x, y]/(y^2 - P(x))$. Prove that $\Omega^1_{X/K}$ is freely generated by dx/y. (Hint: it suffices to check that dx/y is a nowhere vanishing section of $\Omega^1_{X/K}$. Treat the points where $y = 0$ and $y \neq 0$ separately.)

(b) Prove that $H^1_{\mathrm{dR}}(X)$ admits the basis

$$x^i \frac{dx}{y} \qquad (i = 0, \dots, 2g - 1).$$

(Hint: for each integer $d \geq 2g$, write down a relation of the form $Q(x)dx/y$ with $\deg(Q) = d$.)

(c) Let Y be the affine scheme $\operatorname{Spec} K[x,y,z]/(y^2 - P(x), yz - 1)$. Prove that $H^1_{\mathrm{dR}}(Y)$ admits the basis

$$x^i \frac{dx}{y}, \qquad (i = 0, \dots, 2g-1); \qquad x^i \frac{dx}{y^2} \qquad (i = 0, \dots, 2g).$$

(5) Let $p > 2$ be a prime. Let $\overline{P} \in \mathbb{F}_p[x]$ be a monic polynomial of degree $2g+1$ with no repeated roots.

(a) Put $\overline{X} = \operatorname{Spec} \mathbb{F}_p[x,y]/(y^2 - \overline{P}(x))$. Prove that $H^1_{\mathrm{MW}}(\overline{X})$ admits the basis

$$x^i \frac{dx}{y} \qquad (i = 0, \dots, 2g-1).$$

(b) Put $\overline{Y} = \operatorname{Spec} \mathbb{F}_p[x,y,z]/(y^2 - \overline{P}(x), yz - 1)$. Prove that $H^1_{\mathrm{MW}}(\overline{Y})$ admits the basis

$$x^i \frac{dx}{y}, \qquad (i = 0, \dots, 2g-1); \qquad x^i \frac{dx}{y^2} \qquad (i = 0, \dots, 2g).$$