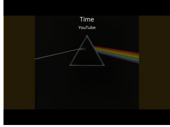
Math 206: Prismatic cohomology



Pink Floyd, *The Dark Side of the Moon* album art

Course web site: https://www.math.ucsd.edu/~kkedlaya/math206/

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Some practical matters

For those watching live:

- These lectures are being recorded and will be posted publicly.
- Unmute is disabled, but I am monitoring the Zoom chat for questions.
- On the web site, you will find a "Miro live" link (one for each lecture) for a browser view of these slides as I write them (in which you can change pages).
- Every lecture will be followed by 30 minutes of office hours in a *different* Zoom meeting. See below.

For those watching the recorded videos:

- I will announce on Zulip when I post a new lecture. This should usually be within 30 minutes of completion.
- The "Miro live" link still works, but use the "boards" link to download the slides as a single PDF.
- I have additional office hours at other times of day. See Zulip.

For everyone:

- Join Zulip! Use the Google Form linked from the course web site. (You can do this anytime during the term.)
- Come to office hours! They will not be recorded. Questions off topic are welcome. See Zulip for the Zoom link.
- For office hours, there is a *shared* Miro whiteboard (à la Google Jamboard). See Zulip for the Miro link.
- Use Zulip to ask (and answer!) questions outside of lecture/office hours. I may follow up on Zulip or in a later lecture.

For UCSD affiliates and other San Diego-based participants:

- You should have an extra stream on Zulip for local announcements. If not, PM me and I'll add you.

About the lecture notes

I am preparing lecture notes for this course. These will be based closely on Bhargav Bhatt's lecture notes from fall 2018, with input also from the Bhatt-Scholze preprint on prismatic cohomology; I will occasionally omit some arguments that can be found in one of those sources (especially later on). I do plan to insert additional background material on some topics as appropriate; requests for topics that could use some background treatment are welcome (e.g., via Zulip).

The sections of the notes correspond *roughly* to individual lectures. As we go along, I may rearrange material a bit to make this correspondence more precise. On the web site, I will track how each lecture maps into the notes.

The notes themselves are available in two different formats: an interactive HTML version and a more standard PDF compiled from LaTeX. Both the HTML and LaTeX code are generated from the same source written in PreTeXt, a markup language built on top of XML; when I make changes (which will happen frequently!) they will usually appear in both formats simultaneously.

The PreTeXt source will be available in a public GitHub repo in case you want to figure out PreTeXt or submit pull requests (e.g., corrections of typos or mathematical errors). **Warning:** I don't plan to push changes to GitHub as often as I make them locally, but I'll try to do it a couple of times per week.

I will do my best to keep the lecture notes at least a week ahead of the lectures, but no guarantees!

And now for some mathematics...

Note: today's lecture is meant to provide a high-level overview of the course material. I am going to omit many definitions and all proofs; you definitely *do not* need to be familiar with everything I am describing in order to follow the course! (On the other hand, some of this material is not going to be treated in the course.)

I'm also going to skip some things which are written out in the notes. This will happen throughout the course, but usually to a lesser extent.

Homology and cohomology of complex varieties

X = algebraic variety / $\mathbb{C}$
(say, smooth, projective)

singular (Betti) homology/cohomology
depends only on X^{an} as top. space, has $\mathbb{Z}$-coefficients.

cohomology of differential forms
(de Rham)

Comparison: Betti vs. de Rham

c, ω $\longrightarrow$ $\int_c \omega$

cycle, differential

Stokes: $\int_{\partial c} \omega = \int_c d\omega$

get well-defined pairings

$H_i(X, \mathbb{Z}) \times H^i_{dR}(X) \longrightarrow \mathbb{C}$

induces a perfect pairing

$H_i(X, \mathbb{C}) \times H^i_{dR}(X) \longrightarrow \mathbb{C}$.

these both come from Hodge structure

i.e. get a $\mathbb{C}$-vector space + "extra structure" ($\mathbb{Z}$-structure, Hodge filtration) ...

The trouble with torsion

Hodge structure only retains $H_{\bullet}(X^{an}, \mathbb{Z})$ modulo torsion

e.g. X = Enriques surface (e.g. K3 surface free modulo)

ω_X is nontrivial 2-torsion in $\mathrm{Pic}(X)$

$\rightarrow H_2(X^{an}, \mathbb{Z}/2\mathbb{Z}) \neq 0$ (nontrivial 2-torsion in $H_{\bullet}(X^{an}, \mathbb{Z})$)

Q Is there a way to detect this sort of torsion on the "de Rham side"?

Cohomology for p-adic algebraic varieties

X smooth proper algebraic variety over ~~K~~ $\mathbb{C}_p$. (K finite)

we look at $H^i_{dR}(X)$ ← K-vector space

algebraic de Rham cohomology

(with Hodge filtration)

and ~~singular~~ étale cohomology

with $\mathbb{C}_p$-coefficients

and crystalline cohomology ("char p de Rham")

The role of prisms provide base rings for

Example: integral p-adic Hodge structures

p = prime number

$A = \mathbb{Z}_p[[u]]$ regular 2-dim noetherian local ring

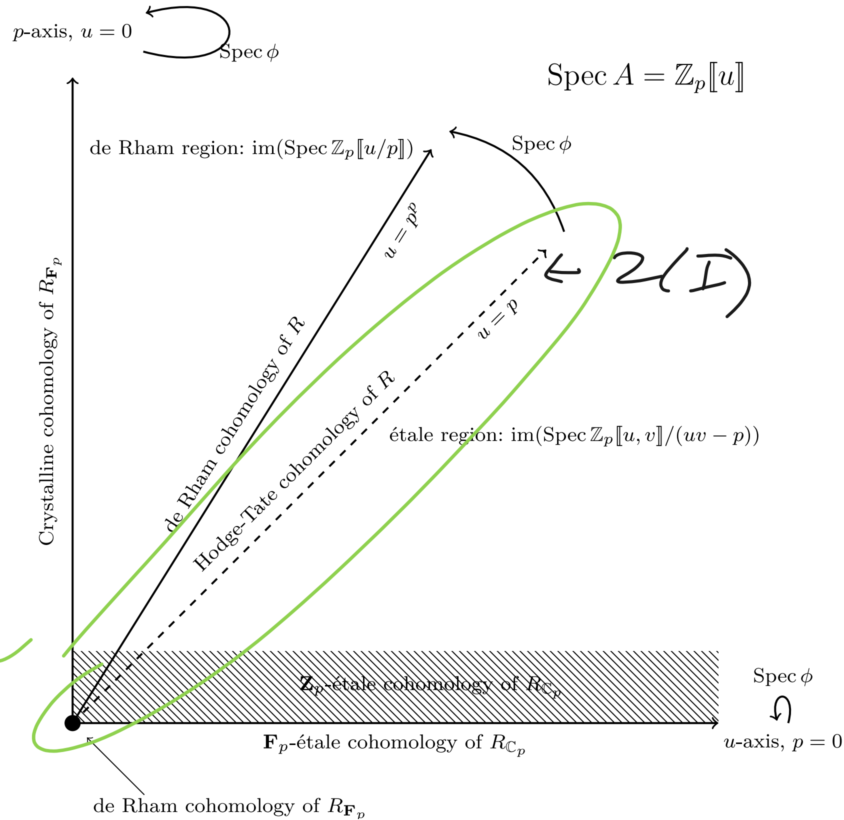
$\phi : A \to A$ $\quad u \mapsto u^p$

$I = (u - p)$ $\quad \theta : A \to A/I \cong \mathbb{Z}_p$

$u \longmapsto p$

A sample theorem: pictorial representation

"original
Spec $\mathbb{Z}_p$"



A sample theorem: statement (part 1)

For R = p-adic completion of a smooth $\mathbb{Z}_p$-algebra

we can functorially 1 ← only in derived category (bounded) define a complex $\Delta_{R/A}$

of (p,u)-adically complete A-modules

and a morphism $\phi_{R/A}: \phi^* \Delta_{R/A} \to \Delta_{R/A}$

such that:

- $\phi_{R/A}$ is a quasi-isomorphism after inverting $u - p$

- can recover other cohomologies by functorial constructions.

A sample theorem: statement (part 2)

e.g. – de Rham cohom of $R\mathbb{F}_p$

– $\mathbb{Z}_p$-étale cohom of $R_{\mathbb{C}_p}$

$$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$$

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