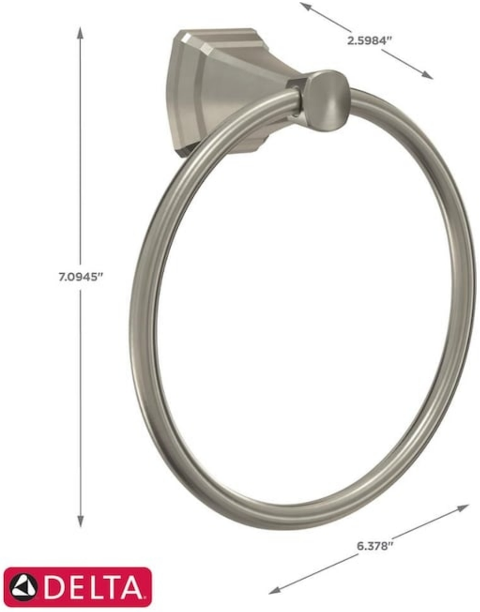
1985 1995

δ-rings (after Joyal and Buium)

I have started marking "unstable" sections of the lecture notes; I suggest not to read these just yet!

While there are no formal assignments, the notes include some exercises. I encourage you to try them and to discuss them in office hours and/or Zulip.



p-derivations on (commutative) rings

p = fixed prime

A δ-ring is a ring A plus a map $\delta : A \to A$
commutative unital
(p-derivation)

s.t.
$$\delta(1) = 0$$
$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\,\delta(x)\delta(y)$$
$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \underbrace{\frac{(p-1)!}{i!\,(p-i)!}}_{\in \mathbb{Z}} x^i y^{p-i}$$

$$\Downarrow$$

$$p\,(\delta(x+y) - \delta(x) - \delta(y)) = x^p + y^p - (x+y)^p$$

$$\Downarrow$$

$$x^p + p\,\delta(x) + y^p + p\,\delta(y) = (x+y)^p + p\,\delta(x+y)$$

p-derivations and Frobenius lifts

Given δ-ring (A, δ), the map $\phi : A \to A$
$\phi(x) = x^p + p\,\delta(x)$

is a ring homomorphism which reduces to Frobenius mod p (i.e. a Frobenius lift)

Conversely, if A is a p-torsion-free ring, then this construction defines a bijection

p-derivations on A $\longleftrightarrow$ Frobenius lifts on A.

δ-structure.

Constant elements of a δ-ring

If (A, δ) is a δ-ring, then

$x \in A$ is $\underline{\delta\text{-constant}}$ if $\delta(x) = 0$

$\Rightarrow \varphi(x) = x^p$

the δ-constant elements form a multiplicative monoid

Examples

- If $p \in A^\times$, then <u>every</u> endomorphism is a Frobenius lift.

- $A = \mathbb{Z}$ has unique Frobenius lift
$\Rightarrow$ unique δ-structure $\delta(x) = (x - x^p)/p$. $\in \mathbb{Z}$

[Note: any δ-structure on <u>any</u> ring has this restriction to $\mathbb{Z}$, even if ring is not p-torsion-free)
in particular, $\delta(p) = 1 - p^{p-1}$

More examples

- $A = \mathbb{Z}[\mu_n : \gcd(n,p) = 1]$

  $\phi : A \to A$ automorphism $\quad \zeta_n \to \zeta_n^p$

  is a Frobenius lift.

  $\Rightarrow$ δ-structure with constants $\bigcup \mu_n$ (plus maybe -1)

  $\gcd(n,p)=1$

- $A = \mathbb{Z}[x]$

  For any $y \in A$, $\exists!$ Frobenius lift ϕ with $\phi(x) = x^p + py$

  $\Rightarrow$ unique δ-structure with $\delta(x) = y$.

  i.e. "set of δ-structures is a free A module

  no module structure

  of rank 1"

What about p-torsion?

Lemma if (A, δ) is a δ-ring and $p^n = 0$ for some $n > 0$, then $A = 0$.

Pf $\delta(p^n) = p^{n-1}\underbrace{(1 - p^{np-n})}_{\text{unit.}}$ $\Rightarrow p^{n-1}$ also zero in A. ...

$0 = \delta(0)$

However you can have examples where $A[p] \neq 0$. easiest to write down using Witt vectors (next lecture).

Lemma if (A, δ) is a δ-ring and $px = 0$ for some $x \in A$, then $\phi(x) = 0$. Hence ϕ injective $\Rightarrow A$ p-torsion-free.

Truncated Witt vectors: set-theoretic definition

A = ring

$W_2(A)$ = set $A \times A$ with binary operations

$$(x_0, x_1) + (y_0, y_1) = \left(x_0 + y_0,\ x_1 + y_1 - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x_0^i y_0^{p-i} \right)$$

$$(x_0, x_1) \times (y_0, y_1) = \left(x_0 y_0,\ x_0^p y_1 + y_0^p x_1 + p x_1 y_1 \right)$$

<u>Lemma</u>: $W_2(A)$ is a commutative ring for these operations

(and $A \mapsto W_2(A)$ is a functor $\underline{\text{Rings}} \to \underline{\text{Rings}}$)

Truncated Witt vectors: ring structure

Pf the map $W_2(A) \to A \times A$ $(x_0, x_1) \mapsto (x_0, x_0^p + px_1)$

is a natural morphism of sets with 2 binary operations

and target is a ring. $\Rightarrow$ $W_2(A)$ is a ring when

A is p-torsion-free, hence in general.

Note: this gives two ring homomorphisms

$W_2(A) \to A$

$G_1(x_0, x_1) = x_0$

$G_2(x_0, x_1) = x_0^p + p x_1$

Truncated Witt vectors and δ-ring structures

A δ-structure on A corresponds to a ring homom.

$w : A \to W_2(A)$ s.t. $\varepsilon_1 \circ w = \mathrm{id}_A$

$w(x) = (x, \delta(x))$

— if A is p-torsion-free
$\operatorname{Spec} W_2(A)$ = two copies of $\operatorname{Spec} A$
glued along $\operatorname{Spec} A/(p)$
via Frobenius

If A not p-torsion-free
this is true in a "derived" sense

The category of δ-rings

A morphism of δ-rings $(A, \delta) \to (A', \delta')$ is a homomorphism $f: A \to A'$ s.t. $f \circ \delta = \delta' \circ f$.

Get a category $\underline{\text{Rings}_\delta}$

<u>Lemma</u> $\underline{\text{Rings}_\delta}$ admits <u>all</u> limits & colimits.

and these commute with $\underline{\text{Rings}_\delta} \to \underline{\text{Rings}}$

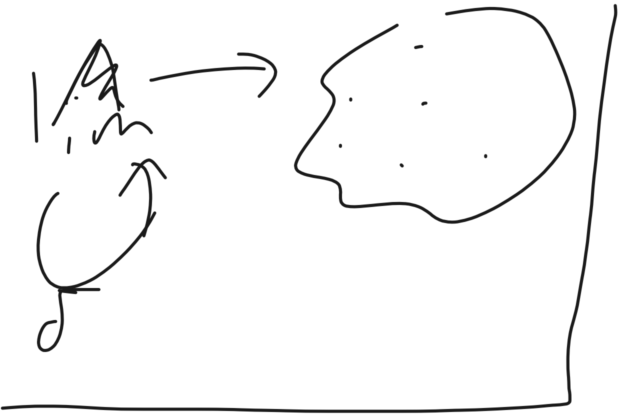
<u>Pf</u> For limits this is straightforward.

For <u>colimits</u>: if A = colimit of diagram $\{A_i\}$ of δ-rings

then $A = \operatorname{colim} A_i \to \operatorname{colim} W_2(A_i) = W_2(\operatorname{Colim} A_i)$

$= W_2(A)$

Induced δ-ring structures (quotients, localizations)



If A = δ-ring

I = ideal s.t. $\delta(I) \subseteq I$

$\Rightarrow$ get induced δ-structure on A/I.

S = multiplicative subset

If $\phi(S) \subseteq S$

$\Rightarrow$ get δ-ring structure on $S^{-1}A$

(reduce to p-torsion-free case)

The left adjoint of the forgetful functor to rings

$\Longrightarrow$ left adjoint + right adjoint.

(adjoint functor/mm)

left adjoint applied to $\mathbb{Z}[S]$

$\mathbb{Z}\{S\} = \mathbb{Z}[S_0 \cup S_1 \cup S_2 \cup \cdots]$

$s_i \in S_i \Longleftrightarrow s_{i+1} \in S_{i+1}$ $\Rightarrow$ $\delta(s_i) = s_{i+1}$.

(it's p-torsion-free!)

Preview: the right adjoint of the forgetful functor

The right adjoint: $\quad$ p-typical witt vectors

will turn out to be $R \longmapsto W(R)$.

Next time: derive <u>all</u> relevant properties of $W(R)$ from this adjunction.

<u>and</u> recover big Witt vectors from there.

δ-rings $\leftrightarrows$ λ-rings