δ-rings (after Joyal and Buium)

I have started marking "unstable" sections of the lecture notes; I suggest not to read these just yet!

While there are no formal assignments, the notes include some exercises. I encourage you to try them and to discuss them in office hours and/or Zulip.
A \text{ \textit{derivation}} \textit{is a ring $A$ plus a map $\delta : A \rightarrow A$ (p-derivation)}

\[ \delta(1) = 0 \]
\[ \delta(xy) = x \delta(y) + y \delta(x) + p \delta(x) \delta(y) \]
\[ \delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i} \in \mathbb{Z} \]

\[ \frac{p}{\delta(xy) - \delta(x) - \delta(y)} = x^p + y^p - (x+y)^p \]
\[ x^p + p \delta(y) + y^p + p \delta(x) = (xy)^p + p \delta(xy) \]
Given a ring \((A, \mathfrak{m})\), the map \(\varphi: A \to A, \varphi(x) = x^p + p \sqrt[p]{x}\) is a ring homomorphism which reduces to Frobenius multiplication \(x \mapsto x^p\) (i.e. a Frobenius lift).

Conversely, if \(A\) is \(\mathfrak{p}\)-torsion-free, then this construction defines a bijection \(p\)-derivations \(\mathfrak{d}\)-structure on \(A\) \(\leftrightarrow\) Frobenius lifts on \(A\).
Constant elements of a $\delta$-ring

If $(A, \delta)$ is a $\delta$-ring, then

$x \in A$ is $\delta$-constant if $\delta(x) = 0$

$\Rightarrow \varphi(x) = x^p$

The $\delta$-constant elements form a multiplicative monoid.
Examples

- If $p \in A^*$, then every endomorphism $f$ of $A^*$ is a fiber, i.e., a lift.

- $A = \mathbb{Z}$ has unique fiberwise 1.17 unique $d$-structure $d(x) = (x - x^2)/p$.

(Note: any $d$-structure on any ring has the restriction to $\mathbb{Z}$, even if $\mathbb{R}$ is not in question, $d(p) = 1 - p^{-2}$.)
More examples

\[ A = \mathbb{Z} \left( \mathbb{Z} : \gcd(n, \rho) = 1 \right) \]

\( \phi : A \rightarrow \mathbb{Z} \) is a homomorphism, \( \overline{g_n} \rightarrow \overline{y^r} \)

\( \Rightarrow \) d-structure with constant \( \mathbb{Z} \) under \( \sum_{\nu \in \mathbb{Z}} (\nu/x^2, \nu) \)

\( \mathbb{A} = \mathbb{Z}(x) \)

For any \( yeA \), \( \exists! \) \( \text{Frobenius lift} \phi \) with \( \phi(x) = x^p + ry \)

\( \Rightarrow \) unique \( d \)-structure with \( \phi(x) = y \).

i.e. "set of \( d \)-structures is a free \( A \)-module"
What about p-torsion?

**Lemma**: If \((A, J)\) is an \(A\)-ring and \(p^n = 0\) for some \(n > 0\), then \(A = 0\).

\[
P + J(p^n) = p^{n-1}(1 - p^m) = 0 \quad \text{for some } m.
\]

However, you can have examples where \(A \neq 0\).

The easiest way to do this is using matrices with vectors.

**Lemma**: If \((A, J)\) is an \(A\)-ring and \(p^n = 0\) for some \(n > 0\), then \(\phi(x) = 0\). Hence \(\phi\) is injective \(\Rightarrow\) \(A\) a p-torsion-free.
Truncated Witt vectors: set-theoretic definition

\[ A = \mathbb{N} \times \mathbb{N} \]

\[ W_2(A) = \text{set } A \times A \text{ with binary operations } \]

\[ (x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1, \frac{(p-1)!}{a_i} x_0^i y_0^{(p-1)-i}) \]

\[ (x_0, x_1) \times (y_0, y_1) = (x_0 y_0, x_0^p y_1 + y_0^p x_1, + p x_1 y_1) \]

Lema: \[ W_2(A) \text{ is a commutative ring for these } \]

\[ \text{operations } (A \times A) \rightarrow W_2(A) \text{ in } \mathbb{R}_{\mathbb{Q}}^{\mathbb{Q}} \]
Truncated Witt vectors: ring structure

\[ f: \mathbb{W}_2(A) \rightarrow A \times A \] \( (x_1, x_2) \rightarrow (x_0, x_0^p + px_1) \)

is a morphism of sets with binary operation natural.

A twist is a ring. \( \Rightarrow \) \( \mathbb{W}_2(A) \) is a ring when \( A \) is properly -free, hence in general.

Note: this gives the ring homomorphism

\[ \mathbb{W}_2(A) \rightarrow A \] 

\[ \varepsilon_1(x_0, x_1) = x_0 \]

\[ \varepsilon_2(x_0, x_1) = x_0^p + px_1 \]
Truncated Witt vectors and δ-ring structures

A δ-structure on A corresponds to a ring homomorphism
\[ w: A \to \mathbb{W}_2(A) \quad \text{s.t.} \quad \omega = \text{id}_A \]
\[ w(x) = (x, \delta(x)) \]

If A is p-torsion-free, \( \text{Spec} \ \mathbb{W}_2(A) \) is the coproduct of \( \text{Spec} \ A \)
given along \( \text{Spec} \ A/k \) via Frobenius.

If A is not p-torsion-free, this is true in a "derived" sense.
The category of δ-rings

A morphism, \( \delta \text{-}	ext{rings} (A, \delta) \to (A', \delta') \)
is a homomorphism \( f: A \to A' \) s.t. \( f \delta = \delta' f \).

Get a category \( \text{Rings} \)

Lemma \( \text{Rings} \) admit all limits & colimits, &

and these commute with \( \text{Rings} \to \text{Rigs} \)

Proof For limits this is straightforward.

For colimits, if \( A \) = colimit of diagram \( \{A_i\}_i \)
then \( \text{colim } A_i \to \text{colim } w_2(A_i) = w_2(\text{colim } A_i) = w_2(A) \)
Induced $\delta$-ring structures (quotients, localizations)

If $A = \mathfrak{f}$-tors.

$\mathfrak{I}'$, ideal s.t. $\mathfrak{I}' \subseteq \mathfrak{I}$

$\Rightarrow$ get induced $\delta$-structure on $A/\mathfrak{I}$.

$S =$ multiplicative subset

If $\phi(s) \subseteq S$

$\Rightarrow$ set $\delta$-ring structure on $S^{-1}A$

(reduce to 0-torsion-free case)
The left adjoint of the forgetful functor to rings

\[ \text{left adjoint} \rightarrow \text{right adjoint}. \]

(left adjoint functor/mon)

left adjoint applied to \( \mathbb{Z}(S) \)

\[ \mathbb{Z}(S) = \mathbb{Z}S_s^1 v_s, v_s \to v_s \ldots \]

\( s_i \vdash s_i \iff s_i \in \mathbb{R} \implies f(s_i) = s_i + 1. \)

(It's a -turn - (nee!))
Preview: the right adjoint of the forgetful functor

The right adjoint will turn out to be \( R \mathcal{C} \rightarrow \mathcal{C} \).

Next time: derive all relevant properties of \( \mathcal{C} \) from this adjunction.

and review big Witt vectors from there.

\( \mathcal{S} \)-rings \( \cong \) \( \mathcal{W} \)-rings