Witt vectors

Office hours for week 2 have been modified; see Zulip. **Important:** no office hours right after Monday's lecture.

In the notes, I separated the p-typical Witt vectors (section 3, this lecture) from lambda-rings and big Witt vectors (section 4, next lecture).
The functor from \( \delta \)-rings to rings has a right adjoint

\[ \text{Ring}_\delta = \text{category of } \delta \text{-rings} \]

\[ \text{set of objects has all limits/colimits} \]

\[ \text{Ring}_\delta \to \text{set} \]

\[ \text{let adjoint \& right adjoint} \]

\[ \text{adjoint functor theorem} \]

*plus a cardinality assertion*

left adjoint gives rise to three \( \delta \)-rings

\[ ( \mathbb{Z}(S) \to \mathbb{Z}(S) ) = \mathbb{Z}(S_0, S_1, S_2, \ldots) \]

right adjoint??
The Joyal coordinate functions

\[ W = \text{right adjoint of } \text{Hom}_{\text{Ring}} \]

\[ W(A) = \text{Hom}_{\text{Ring}}(R \otimes y, W(A)) \]

\[ = \text{Hom}_{\text{Ring}}(12 \otimes y, W(A)) \]

\[ = \{x_0, x_1, \ldots \} \quad \forall i : W(A) \to A \]

\[ = A \times A \times \ldots \]

We will produce a second set of elements

\[ x_0, x_1, \ldots \] (Joyal coordinates)

such that \[ \mathcal{Z}(x_0, x_1, \ldots) = \mathcal{Z}(x_0, x, \ldots) \]

\( \text{(Joyal coordinates)} \)
The Witt coordinate functions \( \phi(y_0) = y_1, \phi(y_1) = y_2 \ldots \)

**Lemma:** In \( \mathbb{Z}/3 \), there exist elements \( x, x, \ldots \)

\[
x_0 = y_0, \quad x_1 = y_1, \quad x_2 = y_2, \ldots
\]

\[
x_n \in y_n + (x_0, \ldots, y_{n-1})Z(x_0, \ldots, y_{n-1})
\]

\[
\phi_n(x_0) = x_0^n + \rho x_1^n + \rho^2 x_2^n + \ldots + \rho^{n-1} x_n
\]

Note: \( \mathbb{Z}/3 \) is skew-free

\[
\phi_m^n(w_0) = w_{0m}
\]
Proof of the lemma

\[
\phi(s) + \cdots + \rho^{n-1} \phi(x_{\lfloor k \rfloor}) = \phi(\phi^{n-1}(x_{\lfloor k \rfloor})) = \phi(x_{\lfloor k \rfloor})
\]

\[
= (x_{\lfloor k \rfloor}^{\rho} + (\cdots) + \rho^{n-2} (x_{\lfloor k \rfloor}^{\rho} + (\cdots) + \rho^{n-1} (x_{\lfloor k \rfloor}^{\rho} + (\cdots) + \rho^{n-2} x_{\lfloor k \rfloor}^{\rho} + \cdots + \rho^{n-1} x_{\lfloor k \rfloor}^{\rho} + \rho^{n-2} x_{\lfloor k \rfloor}^{\rho} + \cdots + \rho^{n-1} x_{\lfloor k \rfloor}^{\rho} + \rho^{n-2} x_{\lfloor k \rfloor}^{\rho}) + \rho^{n-2} x_{\lfloor k \rfloor}^{\rho}) + \rho^{n-1} x_{\lfloor k \rfloor}^{\rho})\]

where \( \ast \in (Y_{1}, \ldots, Y_{n}) \cap (Y_{n+1}, \ldots, Y_{n}) \)

So take \( x_{\lfloor k \rfloor} = s(x_{\lfloor k \rfloor}) + \ast \). These are equal by noting:

\[
s(x_{\lfloor k \rfloor}) = s(Y_{n+1}) + s(Y_{n+2}) + \ast = Y_{n} + \ast
\]
The Witt coordinate functions mod $p$

$(u, \cdot) \in \mathbb{F}(y)$, we have

$$
\phi(x_0) = \phi_0(x_0) = x_0^p \mod p
$$

$$
= \phi_0(\phi_0(x_0)^p + \cdots + x_0^p) \mod p
$$

$$
= \phi_0(x_0^p + \cdots + x_0^p) \mod p
$$

$$
= \phi_0(x_0^p + x_0^p + \cdots + x_0^p) \mod p
$$

$$
= \phi_0(x_0^p + x_0^p + \cdots + x_0^p) \mod p
$$

$$
= x_0^p \mod p
$$

$$
= \phi(x_0) \mod p
$$
Adjunction and the identity map

\[ \forall x \in \text{Mor}_{\text{Ring}}(\mathcal{W}(A), \mathcal{W}(A)) \]

\[ \Delta \in \text{Mor}_{\text{Ring}}(\mathcal{W}(A), \mathcal{W}(\mathcal{W}(A))) \]

\[ \text{diagonal} \]
Constant lifts \( \hat{\mathcal{S}}(Y_2^2) = Y_2^2 \).

\( \Rightarrow \) A relent \( \exists \in \mathcal{W}(A) \) is \( \mathcal{S} \)-constant if

\[ Y_1(2) = Y_2(2) = \cdots = 0. \]

\( \Leftarrow \) \( X_1(2) = X_2(2) = \cdots = 0 \)

call such \( t \) a \( \underline{(\text{constant lift})} \) of \( X_2(2) \in A \).

If \( a \in A \), write \( (a) \) for its constant lift:

Then \( a \rightarrow (a) \) is \( \underline{\text{multiplicative}} \).

Section of \( \Delta_0 : \mathcal{W}(A) \rightarrow A \).
The ghost coordinate functions

\[ \mathfrak{g}_n(x_0) \quad w_n = x_{p+1} + x_1 \]

We can define a related set-theoretic map

\[ U(A) \rightarrow A \times A \times \cdots \]

is not a ring homomorphism unless \( A \) is \( Z_2^{1/2} \) or \( \mathbb{Q} \) and is not injective when \( A \) is a re-union-tree.

(\text{e.g. can use this to show that } \overrightarrow{a} \rightarrow (\overrightarrow{a}, \overrightarrow{a}, \overrightarrow{a}, \ldots) \text{ is multiplicative.})
Applications of the ghost map: Verschiebung, truncations

\[ V(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots) \]

corresponds to ghost side \( \phi \in (0, p\omega, p\omega, \ldots) \)

\[ \text{map is additive, and } \phi \circ V = \mu + \phi. \]

(\text{Note: } \phi \text{ on ghost side is } (\omega^i, \ldots) \rightarrow (\omega, \omega, \ldots) \)

Let \( T \) correspond to \( \omega^i \text{ for } i \in A \).

\[ \text{but } \phi \text{ sends } \omega^i \rightarrow \omega^i (A) \rightarrow \omega^i (A) \]

\( \phi \) does not act on \( \omega^i (A) \) unless \( \nu \rightarrow \nu \) in \( A \).
Witt vectors over a ring of characteristic $p$: Frobenius

If $A$ is a ring of characteristic $p$, then $\phi(x) = x^p$.

$\phi : \mathcal{W}(A) \to \mathcal{W}(A)$

is just $(x_0, x_1, \ldots) \mapsto (x_0^p, x_1^p, \ldots)$.
Witt vectors over a perfect ring of characteristic $p$

If $A$ is perfect of characteristic $(ir. \ x \to x^p$ is surjective)

$\Rightarrow \ \text{w}(A)$ is purity-free $(\text{C} \ \not\vdash \ 1)$

and $p$-adically complete

\( \mathcal{W}(\mathcal{V}) = \mathcal{V} \)

and $f$ is an isomorphism

and $\text{w}(A)/(\mathfrak{p}) \cong A$. 
A universal property of $W(R)$

If $R$ is perfect of characteristic $p$,

$s = p$-adically complete,

then every morphism $R \rightarrow S/\mathfrak{m}$

lifts uniquely to $W(R) \rightarrow s$.

Uses: Every element of $W(R)$ has a unique expansion as $\sum_{n=0}^{\infty} \xi_n \mathfrak{m}^n$.
$p$-adically complete perfect $\delta$-rings are Witt rings

The category of perfect zero-characteristic rings is equivalent to the category of $p$-adically complete, perfect $\delta$-rings via $\mathcal{R} \rightarrow \mathcal{W}(\mathcal{R})$. 