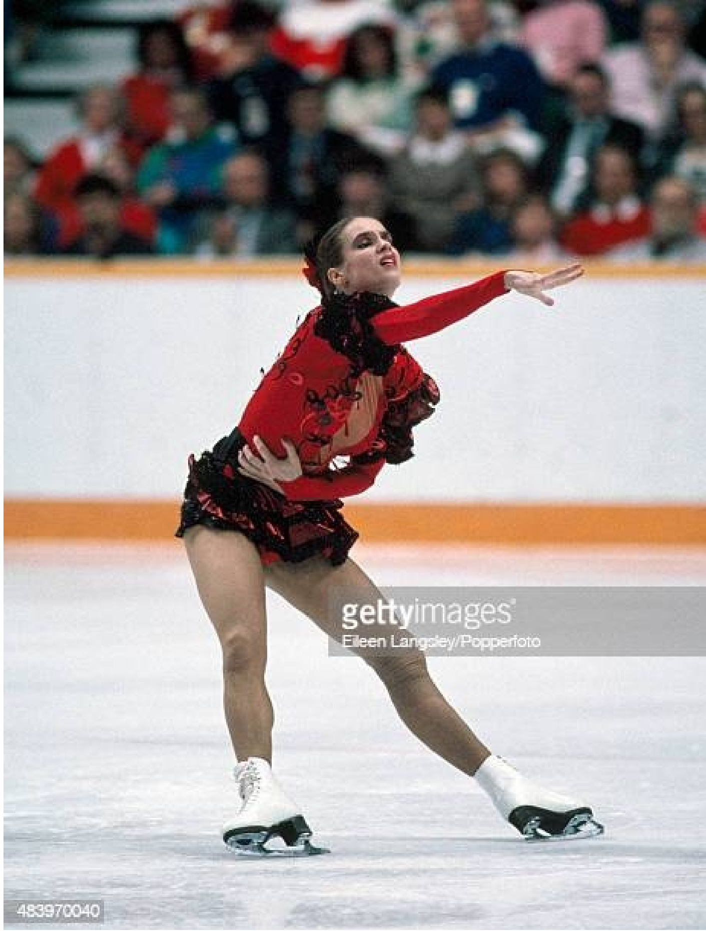
Witt vectors

Office hours for week 2 have been modified; see Zulip. **Important:** no office hours right after Monday's lecture.

In the notes, I separated the p-typical Witt vectors (section 3, this lecture) from lambda-rings and big Witt vectors (section 4, next lecture).



The functor from δ-rings to rings has a right adjoint

$\underline{\mathrm{Ring}_\delta}$ = category of δ-rings

$\downarrow$ forgetful functor has all limits/colimits

$\underline{\mathrm{Ring}}$ * $\Longrightarrow$ get left adjoint & right adjoint

adjoint functor theorem

*plus a cardinality assertion

left adjoint gives rise to <u>free</u> δ-rings

$(\mathbb{Z}[S] \longrightarrow \mathbb{Z}\{S\} = \mathbb{Z}[s_0, s_1, s_2, \ldots]$

$= \mathbb{Z}[s, \delta(s), \delta^2(s), \ldots]$

Right adjoint??

The Joyal coordinate functions

(As sets, we have

W = right adjoint of $\mathrm{Rings}_\delta \to \mathrm{Rings}$)

$$W(A) = \mathrm{Hom}_{\mathrm{Rings}}(\mathbb{Z}[y], W(A))$$
$$= \mathrm{Hom}_{\mathrm{Rings}_\delta}(\underbrace{\mathbb{Z}\{y\}}_{=\mathbb{Z}[y_0, y_1, \ldots]}, W(A))$$
$$= A \times A \times \cdots$$

$y_i : W(A) \to A$

Joyal coordinates

We will produce a second set of elements (with coordinates) $x_0, x_1, \ldots$

such that $\mathbb{Z}[y_0, y_1, \ldots] = \mathbb{Z}[x_0, x_1, \ldots]$

The Witt coordinate functions

$\delta(y_0) = y_1$, $\delta(y_1) = y_2$ . . .

Lemma: In $\mathbb{Z}\{y\}$ there exist unique elements $x_0, x_1, \ldots$

$x_0 = y_0$ $x_1 = y_1$ $x_2 \neq y_2$ . . . and

$x_n \in y_n + (y_1, \ldots, y_n)\mathbb{Z}[y_0, \ldots, y_{n-1}]$

and $\phi^n(x_0) = \underbrace{x_0^{p^n} + p x_1^{p^{n-1}} + p^2 x_2^{p^{n-2}} + \cdots + p^n x_n}$

Note: $\mathbb{Z}\{y\}$ is p-torsion-free

$\Rightarrow$ call this w_n

$\phi^m(w_n) = w_{n+m}$

Proof of the lemma

$x \equiv y \pmod{p^m} \Rightarrow x^p \equiv y^p \pmod{p^{m+1}}$, $m > 0$

$\phi(x_0)^{p^{n-1}} + \dots + p^{n-1}\phi(x_{n-1}) = \phi(\phi^{n-1}(x_0)) = \phi^n(x_0)$

$= (x_0^p + p*)^{p^{n-1}} + \dots + p^{n-2}(x_{n-2}^p + p*)^p + p^{n-1}\phi(x_{n-1})$

$= \underbrace{x_0^{p^n} + \quad + p^{n-2}x_{n-2}^{p^2} + p^{n-1}(x_{n-1}^p}{} + p\delta(x_{n-1})) + p^n *$

where $* \in (y_1, \dots, y_{n-1})\mathbb{Z}[y_0, \dots, y_{n-1}]$

so take $x_n = \delta(x_{n-1}) + *$

these are equal

note: $\delta(x_{n-1}) = \delta(y_{n-1}) + \delta(x_{n-1} - y_{n-1}) + *$

$= y_n + *$

The Witt coordinate functions mod p

Cor: In $\mathbb{Z}\{y\}$, we have $\boxed{\phi(x_n) \equiv x_n^p \bmod p}$

$$x_0^{p^{n+1}} + \dots + p^{n-1} x_{n-1}^{p^2} + p^n x_n^p + p^{n+1} x_{n+1} = \phi^{n+1}(x_0)$$

$$= \phi(\phi^n(x))$$

$$= \phi(x_0)^{p^n} + \dots + p^{n-1}\phi(x_{n-1})^p + p^n \phi(x_n)$$

$$\equiv x_0^{p^{n+1}} + \dots + p^{n-1} x_{n-1}^{p^2} + p^n \phi(x_n) \bmod p^{n+1}$$

$$\Rightarrow p^n x_n^p + \cancel{p^{n+1} x_{n+1}} \equiv p^n \phi(x_n) \bmod p^{n+1}$$

$$\Rightarrow x_n^p \equiv \phi(x_n) \bmod p.$$

Adjunction and the identity map

$$1_A \in \mathrm{Hom}_{Rings}(w(A), w(A))$$

$$X0 = 1/0$$

$$F\,\mathrm{Hom}_{Rings}(w(A), A)$$

$$\neq$$

$$1_A \in \mathrm{Hom}_{Rings}(w(A), w(A))$$

$$\Delta \in \mathrm{Hom}_{Rings}(w(A), w(w(A)))$$

diagonal

Constant lifts

$\delta(y_n) = y_{n+1}$.

$\Rightarrow$) An element $z \in W(A)$ is δ-constant iff

$$y_1(z) = y_2(z) = \cdots = 0.$$

$$\Longleftrightarrow \quad x_1(z) = x_2(z) = \cdots 0$$

call such z a constant lift of $x_0(z) \in A$.

If $a \in A$, write (a) for its constant lift:

the map $a \to (a)$ is a multiplicative

section of $x_0 : W(A) \to A$.

The ghost coordinate functions

$$w_n = \sum_{m \geq 0} p^m x_m^{p^{n-m}}$$

$$w_1 = \cancel{x_0}^{p} + p x_1$$

$\cancel{\phi}_n(x_0)$

The w_n define
a rather self-tread til map
$W(A) \to A \times A \times \cdots$
is not a bijection, but is a ring homomorphism! **ghost map**
(unless A is $\mathbb{Z}(1/p)$-algebra)
but is injective when A is p-torsion-free.
(e.g. can use this to show that
$a \to (a)$ is multiplicative.
or ghost side: $(a, a^p, a^{p^2}, \ldots)$

Applications of the ghost map: Verschiebung, truncations

$V(x_0, x_1, \dots) := (0, x_0, x_1, \dots)$

corresponds on ghost side to $(0, pw_0, pw_1, \dots)$

map is additive, and $\phi \circ V =$ mult by p.

[Note: ϕ on ghost side is $(w_0, w_1, \dots) \to (w_1, w_2, \dots)$

Let truncated Witt vectors $W_n(A)$ for any n. form a ring,

but ϕ sends $W_{n+1}(A) \longrightarrow W_n(A)$

ϕ does not act on $W_n(A)$ unless $p = 0$ in A.

Witt vectors over a ring of characteristic p: Frobenius

If A is char p, $\phi(x_n) \equiv x_n^p \pmod p$ in $\mathbb{Z}[x, y]$

$\phi : W(A) \to W(A)$

~~is~~ is just $(x_0, x_1, \ldots) \to (x_0^p, x_1^p, \ldots)$

Witt vectors over a perfect ring of characteristic p

If A is perfect of char p (i.e. $x \mapsto x^p$ is a bijection)

$\Rightarrow W(A)$ is p-torsion-free ($\Leftarrow \phi$ is injective)

and p-adically complete

($\Leftarrow \phi \circ V = p$ and ϕ is a bijection)

and $W(A)/(p) \cong A$.

A universal property of W(R)

R = perfect of char p

S = p-adically complete

then every morphism $R \to S/(p)$

lifts uniquely to $W(R) \to S$

Uses: Every element of $W(R)$ has a unique "p-adic series expansion" expansion as

$$\sum_{n=0}^{\infty} [x_n] p^n$$

constant lifts

p-adically complete perfect δ-rings are Witt rings

The category of perfect $\mathbb{F}_p$-of-char-p rings is equivalent to category of

p-adically complete, perfect δ-rings (ϕ bijection)

via $\quad R \longmapsto W(R)$.