λ-rings

Reminder: no office hours after today's lecture. See Zulip for this and other schedule adjustments this week.

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Birings

Last time we described the functor W (p-typical Witt vectors) using adjunction to identify the set $W(A)$ with

$$\mathrm{Hom}_{\text{Ring}}(\mathbb{Z}[y_0, y_1, \dots], A)$$
$$= \mathbb{Z}\{y\}$$

In this setup $\mathbb{Z}\{y\}$ is a biring, i.e.

a ring + coaddition $\mathbb{Z}\{y\} \to \mathbb{Z}\{y\} \otimes_{\mathbb{Z}} \mathbb{Z}\{y\}$

& comultiplication $\mathbb{Z}\{y\} \to \mathbb{Z}\{y\} \otimes_{\mathbb{Z}} \mathbb{Z}\{y\}$

+ some conditions

i.e. a commutative ring object in category of affine schemes.

The big Witt vector functor

Prop There is a unique functor $W : \underline{Ring} \to \underline{Ring}$ such that:

1) the underlying functor to sets is
$$W(A) = A_1 \times A_2 \times A_3 \times \cdots$$

2) there is a natural transformation from W to the ordinary product $w : W \to \mathbb{?}^{\mathbb{N}}$ (as $\underline{Rings} \to \underline{Rings}$) given by $(x_n) \mapsto (w_n)$ $\quad w_n = \sum_{d \mid n} d\, x_d^{n/d}$ on sets

$W(A) =$ ring of big Witt vectors over A.

The big Witt vector functor: proof

1) existence/uniqueness clear for $\mathbb{Q}$-algebra (ghost map is a bijection)

2) uniqueness for $\mathbb{Z}$-torsion-free rings $\overset{\text{functoriality}}{\Longrightarrow}$ for all rings.

i.e. I have a ring structure on $\mathbb{Q}[X_1, X_2, \ldots]$ which I want to extend to $\mathbb{Z}[X_1, X_2, X_3 \ldots]$.

well one prime at a time - need to extend to $\mathbb{Z}_{(p)}[X_1, X_2, \ldots]$. Now define $(n \not\equiv 0 \bmod p)$

$Y_n \in \mathbb{Q}[X_1, X_2, \ldots]$ so that $W_{np^i} = \sum_{j=0}^{i} p^j Y_{np^j}^{p^{i-j}}$

claim $\mathbb{Z}_{(p)}[X_1, X_2 \ldots] = \mathbb{Z}_{(p)}[Y_1, Y_2, \ldots]$ $\Leftarrow$ use p-typical Witt vectors.

Additional properties (from the ghost components)

- For any nonempty subset S of $\mathbb{N}$ closed under divisors, set n the factor W_S and projection

$$\begin{tikzcd} W \to W_S & (x_1, x_2, \ldots) \arrow[r, mapsto] \arrow[d] & (x_n)_{n \in S} \arrow[d] \\ & (w_1, w_2, \ldots) \arrow[r, mapsto] & (w_n)_{n \in S} \end{tikzcd}$$

(note: $S = \{1, p, p^2, \ldots\} \to$ p-typical Witt vectors)

- $\phi_n : W \to W$ natural transformation

  on ghost side $(w_1, w_2, \ldots) \mapsto (w_n, w_{2n}, \ldots)$

note: $\phi_n : W_S \to W_{S'}$ if $nS' \subseteq S$

Additional properties (from the ghost components)

- The map $a \mapsto [a] = (a, 0, 0, \dots)$ (constant lift)
  is multiplicative; on ghost side corresponds to $(a, a^2, a^3, \dots)$
- Verschiebung maps $V_n : W(A) \to W(A)$

$$V_n((x_m)) = (y_m) \qquad y_m = \begin{cases} x_{m/n} & m \equiv 0 \pmod n \\ 0 & \text{else} \end{cases}$$

$\phi_n \circ V_n =$ multiplication by n.

- $\Delta : W \to W \circ W$ (diagonal) natural transform,
  characterized by $\Delta([x]) = [[x]]$

Another interpretation via power series

$W(A) \xrightarrow{\sim} 1 + TA[[T]]$ bijection of sets and

$(x_1, x_2, \ldots) \longmapsto \prod_{n=1}^{\infty} (1 - x_n T^n)^{-1}$ homom !!!

additive → multip!

$(x) \longmapsto (1 - xT)^{-1}$

multiplication: $(x)(y) \to (xy)$

$(1 - xT)^{-1} \otimes (1 - yT)^{-1} = (1 - xyT)^{-1}$

say M_1, M_2 = finite projective A mod, $S_i : M_i \to M_i$ A-linear.

$\det(1 - TS_1, M_1)^{-1} \otimes \det(1 - TS_2, M_2)^{-1}$

$= \det(1 - T(S_1 \otimes S_2), M_1 \otimes M_2)^{-1}.$

The definition of a λ-ring

A $\underline{\lambda\text{-ring}}$ is a ring A + operations $\lambda^n : A \to A$ for $n = 0, 1, \ldots$

$\lambda^0(x) = 1$, $\lambda^1(x) = x$

and for $\underline{\Lambda}(x) = (1 - \lambda_1(x)T + \lambda_2(x)T^2 - \ldots) \in 1 + TA[[T]]$

$$\underline{\Lambda}(x+y) = \underline{\Lambda}(x)\underline{\Lambda}(y)$$

$$\underline{\Lambda}(xy) = \underline{\Lambda}(x) \otimes \underline{\Lambda}(y)$$

$$\underline{\Lambda}(\lambda^m(x)) = \Lambda^m \underline{\Lambda}(x)$$

$\Rightarrow$ Category $\underline{\text{Ring}}_\lambda$

Adjoint interpretation of the big Witt functor

The functor W promotes to a functor

$$\underline{\mathrm{Ring}} \to \underline{\mathrm{Ring}},$$

which is a right adjoint of forgetful functor!

(uses $\Delta : W \to W \circ W$)

The Adams operations on a λ-ring

For any λ-ring A ∃ additional maps $\psi^n: A \to A$ (Adams operations)

$$\psi^n \det(1 - TS, M)^{-1} = \det(1 - TS^n, M)^{-1}.$$

in particular $\psi^n([x]) = [x^n]$

so on $W(A)$, $\psi^n = \varphi^n$.

in general, ψ^p is a Frobenius lift, p prime

and these commute pairwise.

λ-rings from commuting Frobenius lifts

If A is $\mathbb{Z}$-torsion-free then

λ-ring structure on A
is equivalent to
a family of pairwise commuting Frobenius lifts.

$\Rightarrow$ In general, a λ-ring is a δ-ring for each p.
(in some compatible way)

Now can show: the elements that are δ-constant for all p are exactly the constant lifts.

Philosophy: What lies below Spec Z?

Idea: (Borger)

$\underline{\text{Rings}_\lambda} \longrightarrow \underline{\text{Rings}}$

$= \text{Rings over } \mathbb{Z} \otimes \mathbb{Z}$

$= \text{Rings over } \mathbb{Z}$

what goes here?

"field of one element"

sphere spectrum