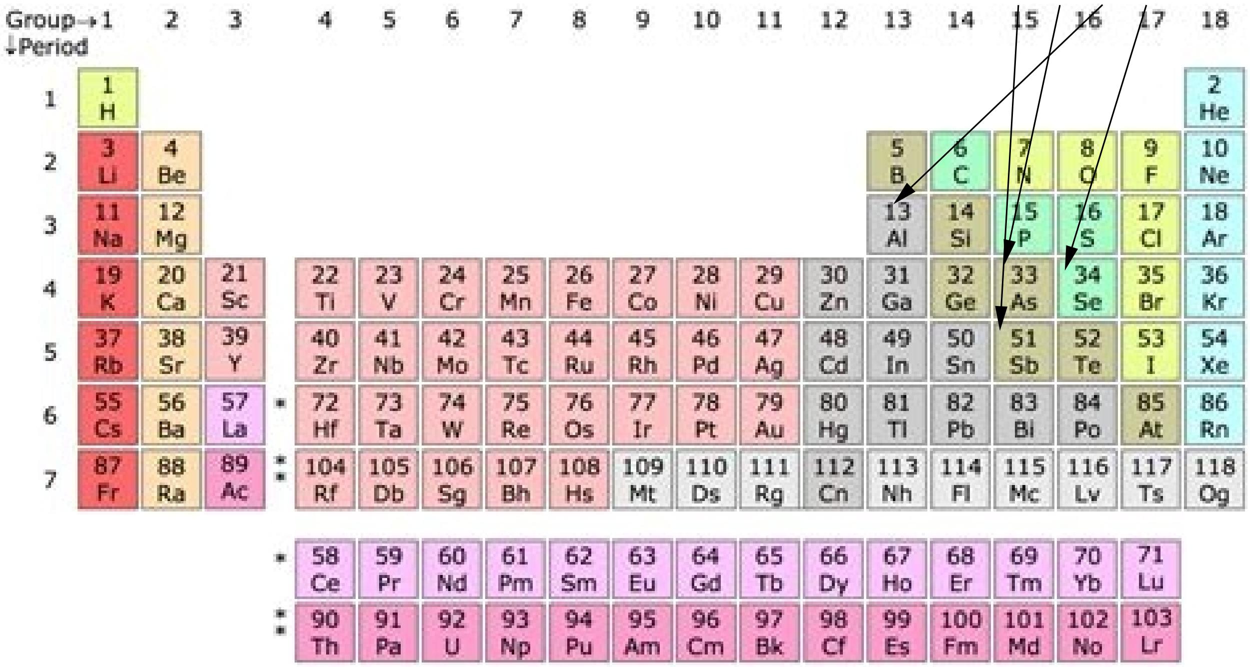
Distinguished elements and prisms

Reminder: see Zulip for schedule adjustments this week.

Some distinguished elements: Sb, As, Al, Se, ...



Distinguished elements of a δ-ring

Fix prime p

Let A be a δ-ring. An element $d \in A$ is __distinguished__ if $(p, d, \delta(d))$ is the unit ideal.

If A is (p,d)-local ($\Leftrightarrow (p,d) \subset \mathrm{Rad}(A)$ = Jacobson radical) this means $\delta(d)$ is a unit.

(Comment: localization of A at (p,d) is still a δ-ring)

If $A \to B$ is a morphism of δ-rings, then distinguished elements go to distinguished elements.

Conversely if $A \to B$ faithfully flat of map is ϕ.

Example: the element *p*

p is distinguished in many ways.

$d(p) = 1 - p^{-1}$

"crystalline"

Example: q-de Rham cohomology (and Wach modules)

$A = \mathbb{Z}_p[[q-1]]$ $\quad \phi(q) = q^p$ $\quad d = [p]_q = \frac{q^p - 1}{q - 1}$

d is distinguished:

$(p, d, \delta(d))$ generate unit ideal: $= \sum_{i=0}^{p-1} q^i$

<u>mod $q-1$</u>, $d \equiv p$ (which is distinguished).

Closely related to Fontaine's theory of <u>(φ, Γ)-modules</u>

(represent p-adic Galois repre~~s~~sentations)

$(\Gamma \cong \mathbb{Z}_p^\times \qquad q \mapsto q^\gamma)$

<u>Wach modules</u> are analogues of B-K modules to this setup.

Example: Breuil-Kisin cohomology

$K/\mathbb{Q}_p$ finite extension $\pi \in K$ uniformizer

$W \subseteq \mathcal{O}_K$ the maximal étale subring over $\mathbb{Z}_p$

$= W(\mathbb{F}_q)$ $A = W[[u]]$ $\phi(u) = u^p$

Consider the map $A \twoheadrightarrow \mathcal{O}_K$ taking $u \mapsto \pi$

d = generator of its kernel $\Rightarrow$ d is distinguished (reduce mod u)

(uses: min poly of π over W is Eisenstein)

B-K modules classify crystalline reps of G_K

Example: A_inf-cohomology

A is a perfect δ-ring

$$A = \mathbb{Z}_p[q^{p^{-\infty}}]^\wedge_{(p,q-1)} \cong W(R)$$

$\phi(q) = q^p$

$R = A/(p) =$ (q-1)-completion of perfection $\mathbb{F}_p[q]$ (colim, iterated ϕ)

$d = (p)_q = \frac{q^p - 1}{q-1}$ is dist.

(via map from $\mathbb{Z}_p[[q-1]]$)

→ makes contact with perfectoid fields/rings

$K = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$ $\quad R = \lim_{\phi} \mathcal{O}_K/(p)$ (tilting)

Irreducibility of distinguished elements

Lemma $A = \delta$-rings, $f, h \in A$, and $f \in Rad(A)$

fh distinguished $\Leftrightarrow$ f distinguished and (p, f, h) is unit ideal.

(if $(p,f) \subset Rad(A)$, says fh distinguished $\Leftrightarrow$ f distinguished & $h \in A^{\times}$)

Key step: $\delta(fh) \equiv h^p \delta(f) \mod (p, f)$

Cor $A = \delta$-ring, $f \in Rad(A)$

f distinguished $\Leftrightarrow$ $p \in (p^2, f, \phi(f))$

[If $p \in Rad(A)$ this becomes $p \in (f, \phi(f))$]

For $\Leftarrow$ direction: assume that $p, f, \delta(f) \subset Rad(A)$

$p = af + b\phi(f) \Rightarrow p(1 - b\delta(f)) = af + bf^p = f(a + bf^{p-1})$

Distinguished elements via their ideals

Lemma: A = p-local δ-ring, I = locally principal ideal contained in $\mathrm{Rad}(A)$

$\Rightarrow$ TFAE: 1. $p \in I + \phi(I)A$

2. I is fpqc-locally generated by a distinguished element

(i.e. $\exists\ A \to A'$ faithfully flat map of δ-rings s.t. $IA' = (d)$ d distinguished.)

Key point: if I is principal, then any gen. is distinguished.

Distinguished elements via their ideals: local version

δ-pairs and prisms

prism is a pair (A, I)

- $A = \delta$-ring
- I = invertible ideal
- A is $\underbrace{\text{derived}}$ (p, I)-complete $\Rightarrow$ A (p, I)-local
- $p \in I + \phi(I)A \Rightarrow I$ locally generated by a distinguished element.

Notes on the definition of a prism

All of my examples of distinguished elements give prisms

Also bounded if A/I has bounded p-power torsion

$$\bigcup_n A/I[p^n]$$
$$=$$
i.e. $A/I[p^\infty] = A/I[p^n]$ for some n.

in this case,
derived (p,I)-complete $=$ (p,I)-complete

Why not classical completeness?

Classical completion behave very badly with respect to homological algebra.

whereas derived (p, I) complete modules form an abelian category.