

Distinguished elements and prisms

Reminder: see Zulip for schedule adjustments this week.

Some distinguished elements: Sb, As, Al, Se, ...

A periodic table of elements with columns labeled Group 1 to 18 and rows labeled Period 1 to 7. The table is color-coded by groups. Four arrows point from the text 'Some distinguished elements: Sb, As, Al, Se, ...' to the elements Antimony (Sb), Arsenic (As), Aluminum (Al), and Selenium (Se) in the table.

Group→	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																	2 He
2	3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	* 72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	* 104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				* 58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				* 90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

Distinguished elements of a δ -ring

(Fix prime p)

Let A be a δ -ring. An element $d \in A$ is distinguished if $(p, d, \delta(d))$ is the unit ideal.

If A is (p, d) -local ($\Leftrightarrow (p, d) \subset \text{Rad}(A) = \text{Jacobson radical}$) this means $\delta(d)$ is a unit.

(Comment: localization of A at (p, d) is still a δ -ring)

If $A \rightarrow B$ is a morphism of δ -rings, the distinguished elements go to distinguished elements.

(Conversely, if $A \rightarrow B$ faithfully flat or map is ϕ .)

Example: the element p

p is distinguished:

"any \mathbb{Z} -Alg

$$d(p) = 1 - p^p \rightarrow$$

"Crystalline"

Example: q -de Rham cohomology (and Wach modules)

$$A = \mathbb{Z}_p \langle q^{-1}D \rangle \quad \phi(\zeta) = q^\zeta \quad d = [p]_q = \frac{q^p - 1}{q - 1}$$

d is distinguished:

$$(p, d, \delta(d)) \text{ generate unit ideal: } = \sum_{i=0}^{p-1} q^i$$

mod $q-1$, $d \equiv p$ (which is distinguished).

Closely related to Fontaine's theory of (φ, Γ) -modules

(represent p -adic Galois representations)

$$(\Gamma \cong \mathbb{Z}_p^\times \quad \varphi: 1 \rightarrow q^\times)$$

Wach modules are analogues of B - K modules
for this setup.

Example: Breuil-Kisin cohomology

K/\mathbb{Q}_p finite extension $\pi \in K$ uniformizer

$W \subseteq \mathcal{O}_K$ the maximal étale subring over \mathbb{Z}_p

$= W(\mathbb{F}_q)$ $A = W[\![u]\!] \quad \phi(u) = u^p$

(consider homomorphism $A \rightarrow \mathcal{O}_K$ taking $u \mapsto \pi$)

$d = \text{generator of its kernel} \Rightarrow d$ is distorted

(use: minpoly of π over \mathbb{Q}_p is Eisenstein (red mod u))

B - K modules classify crystalline reps of G_K

Example: A inf-cohomology

A is a perfect F -ring

$$A = \mathbb{Z}_p [q^{p^{-\infty}}]^{\wedge} \xrightarrow{(p, q-1)} \cong W(\mathbb{Z})$$

$$\phi(q) = q^p$$

$$R = A/(q-1) =$$

= completion of
(perfect) $F_p(q-1)$
? (limit of ϕ)

$$d = (p)_q = \frac{q^p - 1}{q - 1} \text{ is dist.}$$

(via map from $\mathbb{Z}_p((q-1))$)

\rightarrow makes contact with

perfectoid fields/rings

$$K = \mathbb{C}_p(\mu_{p^\infty})^{\wedge} \quad R = \lim_{\phi} \mathbb{C}_k(\mu_p) \text{ (tilting)}$$

Irreducibility of distinguished elements

Lemma $A = \mathcal{D}\text{-rings}$: $f, h \in A$. and $f \in \text{Rad}(A)$

fh distinguished $\Leftrightarrow f$ distinguished and
 (p, f, h) is unit, ideal.

(if $(p, f) \in \text{Rad}(A)$, says fh distinguished $\Leftrightarrow f$ distinguished)

Key step: $\mathcal{D}(fh) \cong h^p \mathcal{D}(f) \text{ mod } (p, f)$ & $h \in A^\times$

Cor $A = \mathcal{D}\text{-rings}$, $f \in \text{Rad}(A)$

f distinguished $\Leftrightarrow p \in (p^2, f, \mathcal{D}(f))$

[If $p \in (p^2, f, \mathcal{D}(f))$
 then is because
 $p \in (f, p(A))$

For \Leftarrow direction: assume that $p, f, \mathcal{D}(f) \in \text{Rad}(A)$

$$p = a + b \mathcal{D}(f) \Rightarrow p(1 - b \mathcal{D}(f)) = a + b f^p = f(a + b f^{p-1})$$

Distinguished elements via their ideals

Lemma: $A = \sqrt{}$ -ring, $I =$ locally principal ideal
contained in $\text{rad}(A)$

$\Rightarrow I \subseteq A \subseteq \hat{A}$: 1. $p \in I + \mathfrak{p}(I)A$

2. I is fpqc -locally generated by
a distinguished element

(i.e. $\exists A \rightarrow A'$ faithfully flat s.t.
 $I A' = (d)$ d distinguished.)

Key point: if I is principal, then any gen. is
distinguished.

Distinguished elements via their ideals: local version

δ -pairs and prisms

Prism is a pair (A, I)

• $A = \mathcal{O}$ -ring

• $I =$ invertible ideal

• A is derived (p, I) -complete \Leftrightarrow A (p, I) -local

• $p \in I + \mathfrak{m}(A) \Rightarrow I$ locally generated
by a distinguished
element.

Notes on the definition of a prism

All of my examples distinguished elements since prisms

Also bounded if A/I has bounded p -power torsion

$$\text{or } \widehat{A/I}(p^\infty) = A/I(p^\infty) \quad \text{for some } n$$

in this case

derived (p, I) -complete = (p, I) -complete

Why not classical completeness?

Classical completions behave
very badly with respect to
homological algebra.

whereas
derived (P, I) complete modules
form an abelian category.