Derived completeness

Classical completions behaving badly

classical I-completion $= \widehat{M}$

A = ring I = finitely generated ideal

$M \in \underline{\mathrm{Mod}}_A$ is classically I-complete if $M \to \lim_n M/I^n M$ is an isomorphism.

($\Rightarrow$ map is injective = M is I-adically separated)

- classically I-complete modules are not an abelian category
- $M \mapsto \widehat{M}$ is not even right exact (but it preserves surjections)
- completion does not preserve flatness.

if I not f.g., then $\widehat{\widehat{M}} \neq \widehat{M}$ in general.

An example of Yekutieli

$A = \mathbb{Z}_p \quad I = (p)$

$M_0 = \{ (x_n)_{n \geq 0} \in \mathbb{Z}_p^{\infty} : \lim_{n \to \infty} x_n = 0 \}$ $\quad \mathbb{Z}_p\langle T \rangle$

(i.e. $\mathbb{Z}_p\langle T \rangle$)

$M_1 = \{ (x_n) \in M_0 : x_n \equiv 0 \bmod p^n \}$ $\quad \mathbb{Z}_p[[pT]]$

$M_2 = \{ (x_n) \in M_1 : \lim_{n \to \infty} x_n/p^n \to 0 \}$ $\quad \mathbb{Z}_p\langle pT \rangle$

The modules M_0, M_1, M_2 is I-complete, as are M_0/M_1, M_1/M_2.

but not M_0/M_2 : closure of M_2 in M_0 is M_1.

completion takes

$$0 \to M_2 \to M_0 \to M_0/M_2 \to 0$$

to

$$0 \to M_2 \to M_0 \to M_0/M_1 \to 0$$

Derived I-complete modules

A = ring
I = fin gen ideal

$M \in \mathrm{Mod}_A$ is derived I-complete if

$\forall f \in I$, $\mathrm{Hom}_A(A_f, M) = \mathrm{Ext}^1_A(A_f, M) = 0$

Note: since $0 \to A[T] \xrightarrow{\times(1-Tf)} A[T] \to A_f \to 0$, $T \mapsto f$
is a projective resolution,
this is equivalent to (*) $\mathrm{Ext}^n_A(A_f, M) = 0 \;\; \forall n \geq 0$.

note: this condition (*) for given $f \Rightarrow$ (*) for any multiple of f.

More on the definition of derived completeness

<u>Lemma</u> for given M, the set of $f \in A$ for which $(*)$ holds is a radical ideal I of A.

Pf: $f \in I$, $g \in A \Rightarrow fg \in I$.

- $f \in I \Leftrightarrow f^n \in I$ $\quad (A_f = A_{f^n})$
- $f, g \in I$

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{fg(f+g)} \to 0$$

$(f,g) = $ unit ideal $\Rightarrow f+g \in I$.

$\Rightarrow$ need to check $(*)$ only for a set of generators of I, or some ideal with same radical.

Classical vs. derived completeness

Lemma: 1) M classically I-complete $\Rightarrow M$ derived I-complete

2) if M derived I-complete $\Rightarrow M \twoheadrightarrow \lim_n M/I^nM$.

$\Longrightarrow$ classically I-complete = derived I-complete + I-adically separated.

M classically complete

1) $\mathrm{Hom}_A(A_f, M) = \mathrm{Hom}_A(A_f, \lim_n M/I^nM) = 0$ $\forall f \in I$

$\mathrm{Ext}^1_A(A_f, M) \overset{?}{=} 0$ $\quad 0 \to M \to E \to A_f \to 0$

$e_n \in E$ mapping to $f^{-n} \in A_f$, $\delta_n = f e_{n+1} - e_n \in M$

M complete $\Longrightarrow$ can define $e_n' = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \dots$

$\Longrightarrow f e_{n+1}' = e_n'$, $\Longrightarrow$ sequence splits.

Classical vs. derived completeness

2) Check $M \twoheadrightarrow \lim_n M/f^n M$! (exercise: finish from here)

Given $x_0, x_1, \dots \in M$ $\exists x$ st. $x \equiv x_0 + f x_1 + \dots + f^{n-1} x_{n-1}$ (mod f^n).

$$E = M \oplus \Big(\bigoplus_n A e_n\Big) \Big/ (x_n - f e_{n+1} + e_n) \longrightarrow A_f$$

$$e_n \mapsto f^{-n}$$

note: $M/f^n M = E/f^n E$ (snake lemma) $0 \to M \to E \to A_f \to 0$

Extension, by hypothesis, so $\exists x \in M$ st. $x + e_0$ generates a copy of A_f in E.

$x + e_0 = x - x_0 + f e_1 = x - x_0 - f x_1 + f^2 e_2 \dots$

The category of derived complete modules

A = rings
I = fin. gen. ideal

Prop

NAK

1) (Derived Nakayama) M = derived I-complete then $M = 0 \iff M/IM = 0$

2) Inclusion from derived I-complete A-modules to $\underline{\text{Mod}}_A$ has a left adjoint, derived I-completion

3) The full subcategory of $\underline{\text{Mod}}_A$ consisting of derived I-complete modules is abelian, and formation of kernels & cokernels commutes with inclusion.

The category of derived-complete modules

Cor If A is derived I-complete

$\Rightarrow I \subseteq \mathrm{Rad}(A)$ (apply derived Nakayama to $A/(u)$ where $u \in 1 + \mathrm{Rad}(A)$)

$\Rightarrow$ every finitely presented A-module is derived I-complete.

Comment: In earlier example, M_0/M_2 is not I-adically separated but it is derived I-complete.

Derived completion

$I = (f_1, \dots, f_n)$

derived I-completion functor is composition of derived f_i-completion for $i = 1, \dots, n$.

Derived completion: principal case

Lemma 1) derived f-completion functor is

$$M \longmapsto \widehat{M} = \mathrm{Ext}^1_A(A_f/A, M)$$

changed notation

$(0 \to A \to A_f \to A_f/A \to 0)$

2) There is a natural exact sequence

$$0 \to R^1 \lim_n M(f^n] \longrightarrow \widehat{M} \longrightarrow \lim_n M/f^n M \to 0$$

· transition maps $M(f^{n+1}) \to M(f^n)$ is $\times f$.

Cases where derived completions are classical

Lemma A = ring, $I = (f)$ principal ideal
M = A-mod. If M has bounded f^{∞}-torsion
(i.e. $M(f^{\infty}) = \bigcup_n M(f^n) = \text{some } M(f^n)$)

then derived I-completion = classical I-completion.

Corollary: R = perfect ring of char p.
for any $f \in R$, $R(f^{\infty}) = R(f)$, so
R has bounded f^{∞}-torsion as a module over itself.

Pf $f^{p^n} x = 0 \Rightarrow f^{p^n} x^{p^n} = 0 \xRightarrow{R \text{ perfect}} fx = 0$
($\Rightarrow$ perfect prisms are classically complete)

Derived completeness/completion in a derived category

Similarly for $K \in D(A)$ ← derived category of $\underline{\mathrm{Mod}}_A$

K is <u>derived I-complete</u> if $R\mathrm{Hom}_A(A_f, K) = 0$ $\forall f \in I$.

(suffices to check on generators)

Derived Nakayama still holds:

K derived I-complete, $K \otimes^L_A A/I = 0$ in $D(A/I)$

$\Rightarrow K = 0$ in $D(A)$.

- Derived I-complete objects of $D(A)$ form a full triangulated subcategory with a left adjoint (<u>derived I-completion</u>)
- Object K is derived I-complete $\Leftrightarrow$ $H^i(K)$ are derived I-complete $\forall i$.
- $M \in A$, derived I-completion <u>as a module</u> is H^0(derived I-completion of $M[0]$) but <u>there might be other terms</u>.