Perfect prisms

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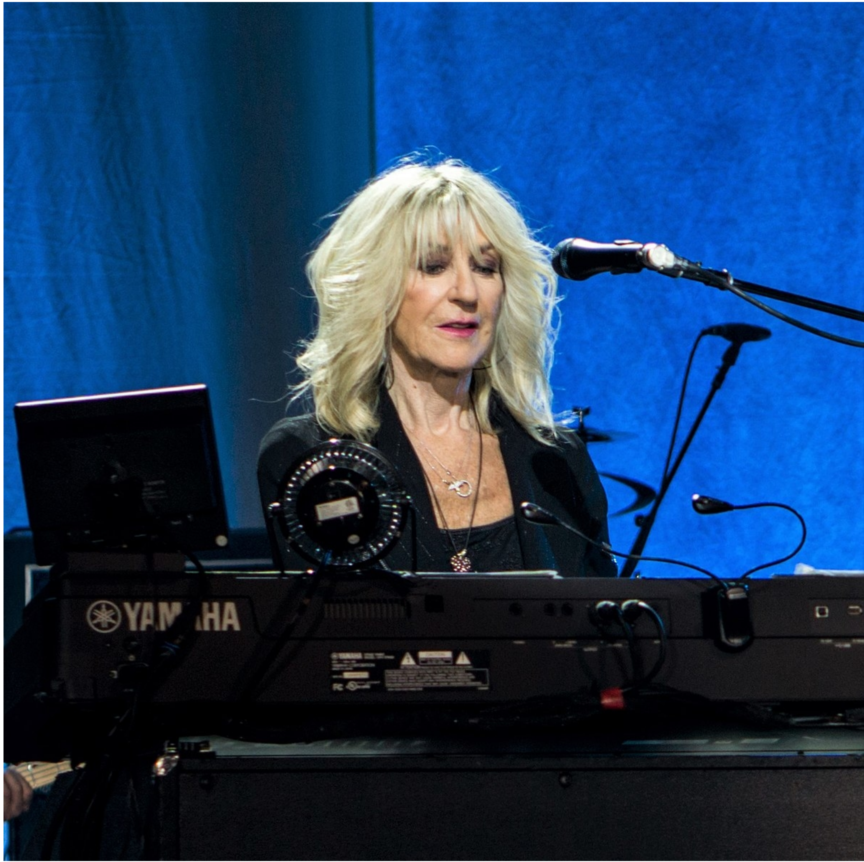

I've added some additional material on derived completeness to section 6 of the notes, most of which I don't plan to cover in lecture.

Schedule note: I need to leave office hours early on Wednesday. If you can save questions for Friday, I'll stay longer to make up for it.

Addendum: derived completion of rings

A = ring, I = finitely generated ideal

$\underline{\mathrm{Mod}}_A \to \underline{\mathrm{Mod}}_A \quad M \longmapsto \widehat{M}$ derived I-completion

induces a functor $\underline{\mathrm{Ring}}_A \to \underline{\mathrm{Ring}}_A$

$B \in \underline{\mathrm{Ring}}_A$

$$\begin{array}{ll} x \mapsto x \otimes 1 & B \longrightarrow B \otimes_A B \\ x \mapsto 1 \otimes x & B \longrightarrow B \otimes_A B \end{array} \qquad \begin{array}{l} B \otimes_A B \xrightarrow{m} B \\ x \otimes y \mapsto xy \end{array}$$

$\widehat{B} \in \underline{\mathrm{Mod}}_A$ derived I-completion as a module

$$\widehat{\widehat{B} \otimes_A \widehat{B}} \longrightarrow \widehat{B \otimes_A B} \xrightarrow{\widehat{m}} \widehat{B}$$

Reminder: the definition of a prism

$A \in \mathrm{Ring}_\delta$

$I =$ ideal of A.

A *δ-pair* is a pair (A, I)

A *prism* is a δ-pair s.t.

- $I =$ invertible A-module.
- A is derived (p, I)-complete over itself

$\Rightarrow (p, I)$-local (derived Nakayama)

- $p \in I + \phi(I)$

$\Rightarrow$ locally I generated by a distinguished element.

(A, I) *orientable* if I principal

oriented if $I = (d)$ with d *chosen*.

(A, I) *bounded* if A/I has bounded p^∞-torsion

Perfect rings, perfection, and coperfection

Ring R in characteristic p is perfect if $\phi : R \to R$ is bijection (i.e. reduced & semiperfect)

A ring morphism is perfect if ϕ is a bijection.

Functor {perfect rings$/\mathbb{F}_p$} $\to$ Rings$/\mathbb{F}_p$

has a left adjoint & right adjoint

$\operatorname{colim}_\phi A = \operatorname{colim}(A \xrightarrow{\phi} A \xrightarrow{\phi} A \dots)$ $\qquad$ $\lim_\phi A = \lim(\dots \xrightarrow{\phi} A \xrightarrow{\phi} A)$

coperfection $\qquad$ perfection

$\mathbb{F}_p(t^{p^{-\infty}}) \leftarrow A = \mathbb{F}_p(t) \longrightarrow \mathbb{F}_p$

Distinguished elements in Witt rings

Reminder: perfect, p-complete δ-rings are all of the form $W(R)$ where R is a perfect ring $/\mathbb{F}_p$.

$d \in W(R)$, $d = \sum_{n \geq 0} [x_n] p^n$ is distinguished iff $(x_0, x_1) = R$. In particular, if $d \in \mathrm{Rad}(W(R))$ (i.e. $x_0 \in \mathrm{Rad}(R)$) then d is distinguished $\iff$ $x_1 \in R^\times$.

(Pf: $d^p \equiv [x_0]^p \bmod p^2$ so $p\delta(d) = \varphi(d) - d^p \equiv p[x_1]^p \pmod{p^2}$)

when to a Weierstrass prepared element of degree 1.

(Fontaine: primitive element of degree 1)

Cor: d is a nonzerodivisor, and $(A/d)[p^\infty] = (A/d)[p]$ if $d \in \mathrm{Rad}(R)$ is dist.

The structure of perfect prisms

(A, I) = perfect prism

Thm • I is principal, generated by a distinguished element d which is not a zero-divisor

• A is p-torsion-free & classically (p, I)-complete.

$\cong W(A/p)$

• $(A/I)[p^\infty] = A/I[p]$, so (A, I) bounded.

Pf: For any prism $\phi(I)A$ is principal

$\Rightarrow$ perfect prism, I is principal

$$\left[\begin{array}{l} p = a + b \\ \quad\;\; \hat{I}A + \phi(I)A \\ \text{show } b \text{ generates } \\ \qquad\qquad \phi(I)A. \end{array}\right]$$

$A/(p)$ is perfect & derived I-complete

$\Rightarrow$ classically I-complete. Devissage $\Rightarrow A/(p^n)$ also classically I-complete

$\Rightarrow A$ is classically (p, I)-complete.

The coperfection of a prism

Inclusion of perfect prisms $\to$ Prism

has a left adjoint (coperfection)

namely: $A \to$ coperfection of A

$\to$ classical (p, I)-completion.

e.g.

$A = \mathbb{Z}_p[[u]] \longrightarrow \mathbb{Z}_p[[u]][u^{1/p}, u^{1/p^2}, \dots]$

$u \to u^p$

$\to$ classical (p, I)-completion

A diagram of a perfect prism

e.g. $\overline{A}/p = \mathbb{F}_p[t^{p^{-\infty}}]/(t)$

$\overline{A}^\flat = \mathbb{F}_p[[t^{p^{-\infty}}]]$

(A, I) = perfect prism

since $\overline{A} = A/I$

$\overline{A}^\flat$ = perfection of $\overline{A}/p$

= tilt of $\overline{A}$

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\sim"] \arrow[dd] & & \overline{A} \arrow[dr, equal, "p\text{-complete}"] \arrow[dd] & \\
& W(\overline{A}^\flat) \arrow[rr, "\theta_A"] \arrow[dd] & & \overline{A} \arrow[dd] \\
A/p \arrow[rr] \arrow[dr, "\sim"] & & \overline{A}/p \arrow[dr, equal] & \\
& \overline{A}^\flat \arrow[rr] & & \overline{A}/p
\end{tikzcd}
$$

$\overline{A} = A/I$

$\overline{A}/p = A/(p, I)$

Recovering a perfect prism from its slice

Recall, n, m: $A \cong W(\bar{A}^\flat)$

$$\bigcap$$

$A/p \cong \bar{A}^\flat$ $\qquad I = (d)$

$\bar{A} = A/I$

$A/(p,d) = \bar{A}/p$

n-fold Frobenius.

$$\begin{tikzcd} A/(p,d) \arrow[r, "\varphi^n"] \arrow[d] & A/(p,d) \arrow[d] \\ A/(p,d^{p^n}) \arrow[r] & A/(p,d) \end{tikzcd}$$

Tilts, untilts, and lenses

<u>Thm</u> Slice functor $(A, I) \to \bar{A}$

is <u>fully faithful</u>, perfect

ie. can recover a prism from its slice.

A <u>lens</u> is a ring, which is a slice of a perfect A prism.

$\{\text{perfect } A \text{ prisms}\} \xrightarrow{\sim} \{\text{lenses}\}$ <u>rings</u>

slice functor. ↑ similar to (integral) <u>perfectoid rings</u>