Lenses

Schedule note: I will need to end the office hour after this lecture a bit early. I'll make up the time on Friday.

Note on the notes: I have split the former section on perfect prisms into two, with the second half becoming the basis for today's lecture.

Some applications of lenses (but not these)
Reminders about perfect prisms

\[ (A,I) = p_{\text{perfect prism}} \]

- \( A = \mathcal{W}(\overline{A^6}) \), \( \overline{A^6} \) is some perfect of characteristic \( p \).
- \( I = (d) \) generated by an distinguished element.
- \( X = \sum (\chi \overline{A^6}) \Rightarrow X \in (\overline{A^6})^* \)
- \((A/I)^{(p)} = (A/I)^{(p)} \cap (A/I) \subseteq (A/I) \cap (A/I) \text{ banded.} \)
Reminder: a diagram of a perfect prism

$\overline{\mathbb{A}} = A/I$ (since $\mathbb{A}$ is a perfect field)

$\overline{\mathbb{A}}^b = 1 \mapsto \overline{\mathbb{A}}/p(-\text{support of } \mathbb{A}(p, d))$

Points in $\mathbb{R}^d$ can be drawn from $\mathbb{A}$

$\widetilde{\mathbb{A}} = \mathbb{A}$
The category of lenses

A lens is a 3-sg which is a slice of the presheaf of rings consistency theories. The full subcategory of rings consisting of lenses is equivalent to perfect presheaves in slicing.

\[ \tilde{A} = \frac{A}{I} \]

for perfect presheaves \[ \tilde{A}^b = \text{hull of } \tilde{A} \]

also say that \( \tilde{A} \) is a until \( A \)

P.S. Consequently perfect presheaves have form \( \text{Hom}(M(R), x) \)

so since \( 3 \leq 12 \), which is also the hull.
Examples

\[ R = t - 2d^2 \text{ complex } \mathbb{C} \Rightarrow \mathbb{F}_{p^2} \]

\[ A = \mathbb{R}, \quad I = (\mathfrak{p}) \quad \longrightarrow \quad \bar{A} = \mathbb{R}(\mathfrak{p}) \]

\[ I = (d) \quad d = \sum_{i=0}^{n-1} t^i \quad \longrightarrow \quad \bar{A} = \mathbb{Z}[t][t^{-1}] \]

\[ I = (\mathfrak{a}) \quad d = \mathfrak{a} - (t) \quad \longrightarrow \quad \bar{A} = \mathbb{Z}[t][t^{-1}] \]
Some intrinsic properties of lenses

Let $R$ be a lens

* $R/R$ is simple and (an injective).

* For every $f: R \to R$, there is a compatible system of corepresentability $(\varphi_f R)$ such that

$$
\varphi = \varphi_f \text{ for } R \to \varphi_f
$$

such that $\varphi$ is generated by $\varphi$.

* $\sqrt{R}$ is ascending union of $R$-lens, i.e., for $\sqrt{R}$ and $\sqrt{R}$

* $R(\varphi) = R(\sqrt{R})$. 

An intrinsic characterization of lenses

Proof: A ring \( R \in \mathbb{R}_m \) is a lens iff:

1) \( R \) is classically p-complete & \( R/p \) is semi-prime.

2) \( \Theta_R : W(12^6) \to R \) has principal kernel

\[
R = \lim_{\phi} \mathcal{Z}/p
\]

3) \( \exists \gamma \in R \) s.t. \( \Theta_p \gamma \neq 0 \) for some \( u \in \mathcal{R} \).

Proof "if": Claim \( A = W(12^6) \) & \( I = \ker(\Theta_R) \) is a prism

(Note: \( R/p \) semi-prime \( \Rightarrow \) \( \mathcal{Z}/p \) structure)

Pick \( x, \forall \mathcal{A} \in \mathcal{L} \) s.t. \( g = p \cdot x \neq 0 \in \ker(\Theta_R) \mathcal{W}(12^6) \)

Observe that \( g \) is distinguished. \( \Rightarrow g \) semi-prime.
Another characterization of $p$-torsion-free lenses

Assume $12 \mathcal{R} \text{sgn} \ is \ p$-torsion-free.

Then $R$ is a lens if:

1. $R$ classically $p$-complete and $R/p$ semi-perfect.
2. $R \text{ is } p$-normal: for any $x \in R(p)$ and $r \in R$.
3. For each $x \in R \text{ P} = R \text{ for some } u \in R^* \text{ and to check:}$ this implies $n \in \text{ker}(R^t) \rightarrow R$ is p-nilpotent.
Examples from perfectoid fields

A perfectoid field is a field $K$ s.t.

- $K = 	ext{w.r.t. to topology induced by some nonarchimedean absolute value}$

- $\mathcal{O}_K/\mathfrak{p}$ is semiperfect

In this line, $\mathcal{O}_K$ is a lens.

Then (T.1) thus for perfectoid fields, after a generalized field automorphism $s$, if $K$ is a perfectoid field, every finite extension $L/K$ s.t. $L:K = [L^b:K^b]$, $K^b = Fr_m(QK^b)$, $G_K = G_{K^b}$
A glueing square for perfect $F_\mathbf{p}$-algebras

Lemma 1: \( R \subseteq R[J] \) is perfect.

\( J = \text{radical ideal of } R, \quad J' = R[C,J] \)

Then \( J' \) is also a radical ideal, and this square is pullback/\( I \)-quotient.

Proof: \( J, J', J + J' \) radical

\( J \cap J' = 0 \)

\( x^2 = 0 \implies x = 0 \)

\[
\begin{array}{ccc}
R & \longrightarrow & R/J' \\
\downarrow & & \downarrow \\
R/J & \longrightarrow & R/(J + J')
\end{array}
\]
A glueing square for lenses

\[ R = \text{len} \quad \overline{R} = \frac{R}{\sqrt{PR}} \]

\[ S = \frac{R}{R(\sqrt{PR})} \quad \overline{S} = \frac{S}{\sqrt{S}} \]

Then all are lenses

\[ \overline{R} \rightarrow \overline{S} \]

1. \( S \) is projection-free
2. \( \overline{PR} \rightarrow \overline{PS} \) is a bijection
3. \( R \rightarrow \overline{R} \) induces a morphism \( R(\sqrt{PR}) \rightarrow \text{ker}(R \rightarrow \overline{S}) = \)

\[ \overline{R} \rightarrow \overline{S} \]

\[ \overline{R} \rightarrow \overline{S} \]

This square is pullbacked
A glueing square for lenses

\[ w(R^b) \rightarrow w(R^b \cap J') \quad \downarrow \quad w(R^b \cap J) \rightarrow w(R^b \cap (J+J')) \]

\[ \begin{array}{ccc}
R & \rightarrow & S \\
\downarrow & & \downarrow \\
\overline{R} & \rightarrow & \overline{S}
\end{array} \]
Completed tensor products

$A \leftarrow B \rightarrow C$ are morphisms of the derived completion of $B \otimes_A C$.

It is concentrated in degree 0 and is a limit.

(Also, it has all colimits, products and equalizers.)