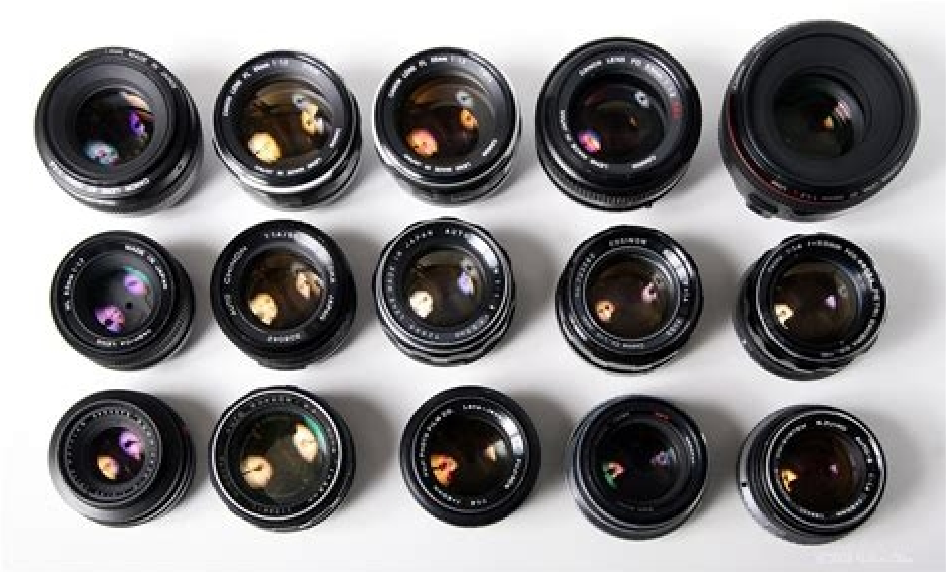
Lenses

Schedule note: I will need to end the office hour after this lecture a bit early. I'll make up the time on Friday.

Note on the notes: I have split the former section on perfect prisms into two, with the second half becoming the basis for today's lecture.



Some applications of lenses (but not these)

Reminders about perfect prisms

(A, I) = $\underbrace{\text{perfect prism}}_{\varphi : A \to A \text{ bijective}}$

- $A = W(\bar{A}^\flat)$ for $\bar{A}^\flat$ = some perfect of char p.
- $I = (d)$ generated by distinguished element
  $d = \sum [x_n] p^n \Rightarrow x_0 \in (\bar{A}^\flat)^*$
- $(A/I)[p^\infty] = (A/I)[p] \Rightarrow (A, I)$ bounded.

Reminder: a diagram of a perfect prism

$\overline{A} = A/I$ (special fiber)

$\overline{A}^\flat = \lim_{\varphi} \overline{A}/p$ (= semiperfect)

$= A/(p,d)$

points on the curve

on the diagram from

$A/I = \overline{A}$

$$
\begin{tikzcd}
A \arrow[rr, two heads] \arrow[dr, "\sim"] \arrow[dd] & & \overline{A} \arrow[dr, equal] \arrow[dd] & \\
& W(\overline{A}^\flat) \arrow[rr, two heads, "\theta_A"] \arrow[dd] & & \overline{A} \arrow[dd] \\
A/p \arrow[rr, two heads] \arrow[dr, "\sim"] & & \overline{A}/p \arrow[dr, equal] & \\
& \overline{A}^\flat \arrow[rr, two heads] & & \overline{A}/p
\end{tikzcd}
$$

The category of lenses

A <u>lens</u> is a ring which is a slice of some perfect prism

The full subcategory of <u>Rings</u> consisting of lenses is equivalent to <u>perfect prisms</u> via slicing

$\bar{A} = A/I$

(A, I) = perfect prism $\bar{A}^\flat$ = <u>tilt</u> of $\bar{A}$

also say that $\bar{A}$ is an <u>untilt</u> $A^\flat$

e.g. crystalline perfect prism has form $(W(R), p)$ so slice is R, which is also the tilt.

Examples

$R = t$-adic completion $\mathbb{F}_p[t^{p^{-\infty}}]$

$A = W(R)$, $I = (p) \rightsquigarrow \bar{A} = R$.

$I = (d)$ for $d = \sum_{i=0}^{p-1} [t+1]^i \rightsquigarrow \bar{A} = \mathbb{Z}_p[\mu_{p^\infty}]^\wedge_{(p)}$

$I = (d)$ $d = p - [t] \rightsquigarrow \bar{A} = \mathbb{Z}_p[p^{p^{-\infty}}]^\wedge_{(p)}$

Some intrinsic properties of lenses

Let R be a lens

* R/p is semiperfect ($\neq$ surjective)
* $\exists$ an element $\varpi \in R$ admitting a compatible system of p-power roots ($\varpi^{p^{-n}}$) s.t.

$\varpi = pu$ $u \in R^*$ and kernel of $\varphi : R/p \to R/p$ is generated by $\varpi^{1/p}$ $\left[d = \underbrace{(a_0)}_{\varpi} = pu \right]$

* radical $\sqrt{pR}$ is ascending union of principal ideals and $(\sqrt{pR})^2 = \sqrt{pR}$ (and $\sqrt{pR}$ is flat over R).
* $R(p) = R(\sqrt{pR}]$.

An intrinsic characterization of lenses

<u>Prop</u> A ring $R \in \underline{\mathrm{Ring}}$ is a lens <u>iff</u>

1) R is classically p-complete & R/p is semiperfect.

2) $\theta_R : W(R^\flat) \longrightarrow R$ has principal kernel

$R^\flat = \lim\limits_{\phi} R/p$

3) $\exists\, \varpi \in R$ s.t. $\varpi^p = pu$ for some $u \in R^\times$.

Proof "if": claim $A = W(R^\flat)$ $I = \ker(\theta_R)$ is a prism

(note: R/p semiperfect $\Rightarrow$ θ_R surjective)

Pick $x, v \in A$ lifts ϖ, u, $g := pv - x^p \in \ker(\theta_R)$

observe that g is distinguished. $\Longrightarrow$ g generates $\ker(\theta_R)$

Another characterization of *p*-torsion-free lenses

Assume $R \in \underline{\mathrm{Ring}}$ is p-torsion-free

Then R is a lens iff:

1. R classically p-complete and R/p semiperfect.
2. R is $\underline{p\text{-normal}}$: Any $x \in R[p^{-1}]$ with $x^p \in R$ is in R.
3. $\exists\, \varpi \in R$ s.t. $\varpi^p = pu$ for some $u \in R^*$

Need to check: this implies $\ker \theta_R : W(R^\flat) \longrightarrow R$ is principal.

Examples from perfectoid fields

A perfectoid field is a field K s.t.

- K = complete for topology induced by some nonarchimedean abs. val with nontrivial & nondiscrete value group.
  or residue char p
- $\mathcal{O}_K/p$ is semiperfect.

In this case, $\mathcal{O}_K$ is a less!

Thm (Tilting for perfectoid fields aka generalized Fontaine-Wintenberger field of norms)
(Kedlaya-Liu, S)

if K is a perfectoid field, every finite extension L of K is too and $[L:K]=[L^\flat:K^\flat]$

$K^\flat = \mathrm{Frac}(\mathcal{O}_K^\flat) \Rightarrow G_K \cong G_{K^\flat}$

A glueing square for perfect F_p-algebras

Lemma: $R \in$ Rings$_{\mathbb{F}_p}$ perfect.

J = radical ideal of R, $J' := R[J]$

Then J' is also radical ideal, and this square is pullback & pushout.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/J' \arrow[d] \\ R/J \arrow[r] & R/(J+J') \end{tikzcd}$$

Pf $J, J', J+J'$ radical

$J \cap J' = 0$

$x \in$ $\Rightarrow xJ = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$.

A glueing square for lenses

R = lens $\quad \overline{R} = R/\sqrt{pR}$

$S = R/R(\sqrt{pR}) \quad \overline{S} = S/\sqrt{pS}$

Then all are lenses

and

this square is pullback & pushout.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ \overline{R} \arrow[r] & \overline{S} \end{tikzcd}$$

and

1. S is p-torsion-free
2. $\sqrt{pR} \to \sqrt{pS}$ is a bijection
3. $R \to \overline{R}$ induces an isomorphism
   $R(\sqrt{pR}) \cong \ker(\overline{R} \to \overline{S}) \Rightarrow$

Frobenius is a bijection on $R(\sqrt{pR})$.

[so any lens is reduced.]

A glueing square for lenses

proof: see notes

$$\begin{tikzcd} W(R^b) \arrow[r] \arrow[d] & W(R^b/J') \arrow[d] \\ W(R^b/J) \arrow[r] & W(R^b/(J+J')) \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ \overline{R} \arrow[r] & \overline{S} \end{tikzcd}$$

Completed tensor products

$A \to B$ are morphisms of these
$A \to C$

the derived p-completion of $B \otimes_A C$

is concentrated in degree 0

and is in these.

(also have all colimits, products
(but not equalizers))