Homotopy categories and derived categories

Note: this lecture covers two sections of the notes (9 and 10). There is additional material (especially in section 10) which I'm not (currently) planning to cover in lecture.
Motivation: right derived functors

\[ F : A \to A', \text{ a right exact functor between abelian categories} \]

If \( 0 \to M_1 \to M_2 \to M_3 \to \cdots \) is exact, then \( \text{coker} \, F(M_i) \to F(M_{i+1}) \to F(M_{i+2}) \to \cdots \)

is a sequence of exact sequences. Assume \( A' \) has enough injectives.

Define right derived functors \( R^i F : A \to A \)

\[ R^0 F = F \]

\[ R^1 F(M) = \text{coker} \, F(M) \to F(M_1) \to F(M_2) \to F(M_3) \to \cdots \]

Note, \( R^i F(M) \) can be thought of as the resolution \( J \) of \( M \).

\[ 0 \to M' \to J_0 \to J_1 \to \cdots \]
The category of chain complexes

A chain complex in $A$ is a sequence

$$A_0 \longrightarrow A_1 \longrightarrow \cdots$$

s.t. $d^n \circ d^{n+1} = 0$.

Differential

$D: (\text{unbounded below})$ $\text{bounded above}$. $D$ is a morphism in $\text{Comp}(A)$. $\text{Comp}(A)$ is the category of chain complexes, an object is a complex and a morphism is a zigzag

$$h^n(k^\bullet) \rightarrow h^n(l^\bullet)$$
Split exact sequences

c a c h o i s e s i . t . s . t .
sto t + s d s = IA N.

0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0 \text{ is split exact.}

If it's exact and

\[ D \circ f = 1_{IA N} \]

\[ D \circ g = 1_{IA N} \text{ s.t.} \]

\( D \) is the opposite of \( M \).

\( g \circ (D = IA N) \)

\( D \) is the image under any functor is split exact.

\( C(i) : A \to \text{Cov}(A) \quad M \to (\to \text{Cov}(M) \to \text{Cov}(A)) \)

\( C(i) : \text{Cov}(A) \to \text{Cov}(A) \quad K \circ C(i) = K \)

\( C(i) \circ C(i) = C(i \circ i) \)
The homotopy category

\[ f: K_1 \to K_2 \] morphisms in \( \text{Comp}(\mathcal{A}) \).

A hom homotopy for \( f \) is a sequence of morphisms

\[ h_n: K_1 \to K_2, \quad n\in\mathbb{N} + 1, \quad \{ f \}_{n=0}^{\infty} \Rightarrow h(f) = 0 \quad \forall n \]

\[ f^n = d_{K_2}^n h_n + h_{n-1}, \quad n = 2 \]

\( \Rightarrow \) homotopy category \( K(\mathcal{A}) = \text{category with} \)

objects = objects of \( \text{Comp}(\mathcal{A}) \)

morphisms = morphisms in \( \text{Comp}(\mathcal{A}) \) / morphisms homotopic to zero.

\( i_1: K(\mathcal{A}) \to \mathcal{A}, \quad i_0 \circ i_0 = i_1 \mathcal{A} \)
Injective resolutions in the homotopy category

Given $M \in \mathcal{A}$, let $I$, $J$ be injective resolutions.

1. Find a morphism $I \to J$ in $\text{Cay}[\mathcal{A}]$ which commutes with $\text{MC}(\theta) \to \Sigma^\theta$.

2. In $K(\mathcal{A})$, this morphism is unique.

3. Hence $I \to J$ are canonically isomorphic in $K(\mathcal{A})$.

Similarly, given $M, N \in \mathcal{A}$, $M \to N$, let the unique map $\frac{I}{J}$, in $K(\mathcal{A})$,
Derived functors in the homotopy category

\( \Gamma: A \to \mathcal{A} \) left exact below

\( R\Gamma: K^+(A) \to K^+(A) \)

\( (M^\bullet) \xrightarrow{\beta} (I^\bullet) \quad R\Gamma(M^\bullet) = F(I^\bullet) \)

\( R\text{Hom}_A(M, \ast) \) (in \( \text{Mod}_A \))

\( \text{M} \otimes^L_A \ast \)

\( R^iF = H^i \circ R\Gamma \)

If \( M^\bullet \neq I^\bullet \) in \( K(A) \).
Localization in a category

We want to investigate morphisms (small) $C = \text{my category}$

Let $S$ be a collection of morphisms.

1) $S$ contains all identities & closed under composition.

2) $x \to y$, $x \to y'$

3) (E add) for any $f : x \to y$ and $t : x \to x \in S$ the $\exists s \in S : y \to s \circ y', s \circ f = 0$.

Wrtt multiplicative system, multiplicative system.
Mapping cones

\[ f : K^o \to L^o \ni \ker \circ (\mathcal{H}) \]

core is complex \[ C(f) \] \[ C(f) \circ = L^c \circ K^c \circ \]

\[ d^n_{C(f)} = \begin{pmatrix} d^n_1 + t^n & \text{???} \\ -d^n_{K} & \text{???} \end{pmatrix} \]

Set a triangle

\[ K^o \to L^o \to C(f)^{o} \to K^{c}(f) \]

so \( \ker H^{0} \) gives a long exact sequence.

Any \( H^{0} \) isomorphic to one of there are distinguished triangle
Distinguished triangles

A morphism $K' \to L' \to M' \to K' \cdot C D$

is distinguished if it is isomorphic to the

triple at some core.

On triangles, we have operators

$(\mathcal{N})$: preserves distinguished triangles

homotopy: $L' \to M' \to K' \cdot C D \to L' \cdot (C D)$

lemma: this preserves distinguished triangles
The long exact sequence of a distinguished triangle

\[ K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1] \]

A distinguished \( d \).

\[ \Rightarrow \text{exact sequence} \]

\[ \cdots \to H^r(M) \to H^r(K) \to H^r(L) \to H^r(K)[1] \to \cdots \]
Rotations of distinguished triangles
Another lemma about distinguished triangles

\[ A^\circ \xrightarrow{f} B^\circ \]
\[ A^\circ \xrightarrow{f} B^\circ \xrightarrow{c(f)} A(1)^\circ \]
\[ A^\circ \xrightarrow{f} B^\circ \xrightarrow{c(f)} A(1)^\circ \]

\[ A^\circ \xrightarrow{f} B^\circ \xrightarrow{c(f)} A(1)^\circ \]
\[ A^\circ \xrightarrow{f} B^\circ \xrightarrow{c(f)} A(1)^\circ \]

\[ \text{dististinguish } \lambda. \]
Quasi-isomorphisms form a multiplicative system

Use previous slide to show

\[ \text{similar for condition 3} \]
The derived category of modules over a ring

If $A$ is a small abelian category, set

$D(A) = \text{st} K(A)$  \text{st = all quasi-isomorphisms}

(\text{morphisms of cyclic core})

For $A = \text{Mod } A$, can also just truncate.

Grothendieck abelian category