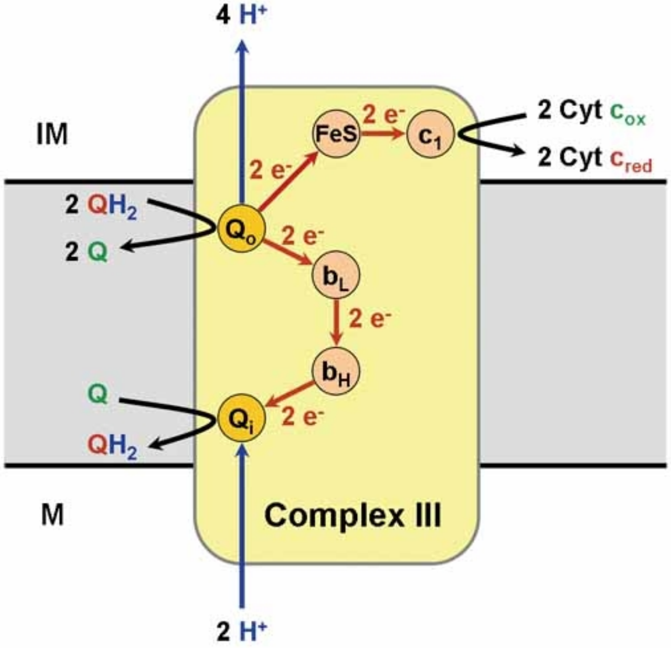
Homotopy categories and derived categories

Note: this lecture covers **two** sections of the notes (9 and 10). There is additional material (especially in section 10) which I'm not (currently) planning to cover in lecture.



Motivation: right derived functors

$\mathcal{A}$ = abelian category

$F : \mathcal{A} \to \mathcal{A}'$ additive left exact functor between abelian categories

If $0 \to M_1 \to M_2 \to M_3$ exact then $0 \to F(M_1) \to F(M_2) \to F(M_3)$.

assume $\mathcal{A}$ has enough injectives

Can define right derived functors $R^iF : \mathcal{A} \to \mathcal{A}'$ $(i = 0, 1, \ldots)$

$R^0F = F$

If $0 \to M_1 \to M_2 \to M_3 \to 0$ exact in $\mathcal{A}$, then $0 \to F(M_1) \to F(M_2) \to F(M_3) \xrightarrow{\delta^0} R^1F(M_1) \to R^1F(M_2) \to R^1F(M_3) \xrightarrow{\delta^1}$

namely, $R^iF(M)$ are cohomology of an injective resolution $I^\bullet$ of M, (i.e. $0 \to M \to I^0 \to I^1 \to \cdots$).

The category of chain complexes

$\mathcal{A}$ = fixed abelian category

A <u>chain complex</u> in $\mathcal{A}$ is a sequence in $\mathcal{A}$

$$\cdots \to K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \to \cdots$$

[<u>cohomological</u> numbering]

s.t. $d^n \circ d^{n-1} = 0 \ \forall n$.

<u>differentials</u>

(bounded below, bounded above, bounded)

$$\boxed{h^n(K^\bullet) = \ker(d^n)/\mathrm{im}(d^{n-1})}$$

$\mathrm{Comp}(\mathcal{A})$ = category of chain complexes

a morphism is a diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & L^{n-1} \arrow[r] & L^n \arrow[r] & L^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

induces maps

$$h^n(K^\bullet) \to h^n(L^\bullet).$$

Split exact sequences

Cnchivse s.t. s.t $f\circ t + s\circ g = \mathrm{id}_N$.

$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is split exact

if it's exact and:

- $\exists\, t: N \to M$ s.t. $t\circ f = \mathrm{id}_M$
- $\exists\, s: P \to N$ s.t. $g\circ s = \mathrm{id}_N$

$\Rightarrow$ image under any functor is split exact.

$(i): A \to \mathrm{Comp}(A)$ $\quad M \to (\cdots \to 0 \to \underset{\text{degree } -i}{M} \to 0 \to \cdots)$

$(i): \mathrm{Comp}(A) \to \mathrm{Comp}(A)$ $\quad K[i]^n = K^{n+i}$.

$(i)\circ(j) = (i+j)$.

The homotopy category

$f: K_1^\bullet \to K_2^\bullet$ morphism in $\mathrm{Comp}(\mathcal{A})$.

A chain homotopy for f is a sequence of morphisms

$$\left.\begin{array}{l} h_n : K_1^n \to K_2^{n-1} \text{ in } \mathcal{A} \text{ s.t.} \\ f^n = d_2^{n-1} \circ h_n + h_{n+1} \circ d_1^n . \end{array}\right\} \Rightarrow H^n(f) = 0 \ \forall n$$

$\Rightarrow$ homotopy category $K(\mathcal{A})$ = category with

objects = objects of $\mathrm{Comp}(\mathcal{A})$

morphisms = morphism in $\mathrm{Comp}(\mathcal{A})$ / morphisms homotopic to zero.

cohomology functors

$H^i : K(\mathcal{A}) \longrightarrow \mathcal{A}$. $\quad H^0 \circ [0] \cong \mathrm{id}_{\mathcal{A}}$

Injective resolutions in the homotopy category

Given $M \in \mathcal{A}$, let $I^\bullet$, $J^\bullet$ be injective resolutions.

1. $\exists$ a morphism $I^\bullet \to J^\bullet$ in $\mathrm{Comp}(\mathcal{A})$ which commutes with

$$
\begin{tikzcd}
M(0) \arrow[r] \arrow[d, equal] & I^\bullet \arrow[d, dotted] \\
M(0) \arrow[r] & J^\bullet
\end{tikzcd}
$$

2. In $K(\mathcal{A})$, this morphism is unique.
3. Hence $I^\bullet$, $J^\bullet$ are canonically isomorphic in $K(\mathcal{A})$.

Similarly, given $M, N \in \mathcal{A}$, $M \to N$

get unique map $I^\bullet \to J^\bullet$ in $K(\mathcal{A})$.

Derived functors in the homotopy category

$F : A \to A'$ left exact

$RF : K^{+}(A) \to K^{+}(A')$ bounded below

$(M^\bullet) \xrightarrow[q.i.s]{} (I^\bullet)$ $\quad RF(M^\bullet) = F(I^\bullet)$

$R\mathrm{Hom}_A(M, *)$ (on Mod_A)

$M \otimes^{L}_{A} *$ on Mod_A, but right exact

$R^i F = H^i \circ RF$

$M^\bullet \to I^\bullet$ is a function

but $M^\bullet \not\cong I^\bullet$ in $K(A)$.

Localization in a category

want to invert quasi-isomorphisms

$$\bullet \longrightarrow \bullet \longleftarrow \cdots \bullet \longrightarrow \bullet \longleftarrow \cdots \bullet$$

$\mathcal{C}$ = any (small) category

S = collection of morphisms

1) S is a

<u>left multiplicative system</u>, if

1) S contains identities & closed under composition.

2)
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "t\in S"'] & Y \\ X' & \end{tikzcd} \Rightarrow \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "t\in S"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, dashed, "s\in S"] \\ X' \arrow[r, dashed, "f"] & Y' \end{tikzcd}$$

(for some Y')

$$\boxed{\rightsquigarrow S^{-1}\mathcal{C}}$$

3) ($\mathcal{C}$ additive) for any $f : X \to Y$ and $t : Z \to X \in S$ s.t. $f \circ t = 0$

then $\exists\, s \in S : Y \to Z$ s.t. $s \circ f = 0$.

right multiplicative system, multiplicative system.

Mapping cones

$f : K^\bullet \to L^\bullet$ in $\mathrm{Komp}(A)$

cone is complex $C(f)^n = L^n \oplus K^{n+1}$

$$d^n_{C(f)} = \begin{pmatrix} d^n_L & f^{n+1} \\ & -d^{n+1}_K \end{pmatrix}$$

Get a triangle $K^\bullet \to L^\bullet \to C(f)^\bullet \to K^\bullet[1]$

s.t. applying H^0 gives a long exact sequence.

Any triangle isomorphic to one of these is

distinguished triangle

Distinguished triangles

A triangle $K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$ is distinguished if it's isomorphic in $K(A)$ to the triangle of some cone.

On triangles, we have operations:

$[n]$: preserves distinguished triangles

__turn rotation__: $L^\bullet \to M^\bullet \to K^\bullet[1] \xrightarrow{(-1)} L^\bullet[1]$

__Lemma__: this preserves distinguished triangles

The long exact sequence of a distinguished triangle

$$K^\cdot \longrightarrow L^\cdot \longrightarrow M^\cdot \longrightarrow K^\cdot(1)$$

distinguished..

$\Longrightarrow$ exact sequence

$$\cdots \to H^n(K^\cdot) \to H^n(L^\cdot) \to H^n(M^\cdot) \to H^{n+1}(K^\cdot) \to \cdots$$

Rotations of distinguished triangles

are distinguished.

Another lemma about distinguished triangles

$$
\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d] & B^\bullet \arrow[d] \\
A'^\bullet \arrow[r, "f'"] & B'^\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C(f)^\bullet \arrow[r] \arrow[d, dashed] & A(1)^\bullet \arrow[d] \\
A'^\bullet \arrow[r, "f'"] & B'^\bullet \arrow[r] & C(f')^\bullet \arrow[r] & B(1)^\bullet
\end{tikzcd}
$$

distinguished.

Quasi-isomorphisms form a multiplicative system

use previous slide to check condition 2

Similar for condition 3

The derived category of modules over a ring

If $\mathcal{A}$ small abelian category, set

$D(\mathcal{A}) = S^{-1}K(\mathcal{A})$ S^{-1} = all quasi-isomorphisms
$\parallel$
(morphisms with acyclic cone.)

$\Downarrow$

For $\mathcal{A} = \operatorname{Mod} A$, can also set this to work.
$\nearrow$
Grothendieck abelian category